Algebra Qualifying Examination

January 1997

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes: All rings are unitary. All modules are unitary. $\mathbf{Q}$ is the rationals, $\mathbf{R}$ the reals, $\mathbf{C}$ the complexes, and $\mathbf{Z}$ the integers.

1. (10pts) Show there is no simple group of order 48.
2. (10pts) Determine the number of 3×3 invertible matrices over a finite field having q elements.
3. (10pts) Let R be a commutative ring. Prove that all R-modules are free if and only if R is a field.
4. (20pts) (a) Show that every ring R has a maximal ideal.
   (b) Give an example of a ring with a unique maximal nonzero ideal.
   (c) Give an example of a ring with a finite (> 1) number of maximal ideals.
   (d) Give an example of a ring with an infinite number of maximal ideals.
   (e) Give an example of a non-commutative ring in which $\{0\}$ is the unique maximal ideal.
5. (15pts) Let K be the splitting field of the polynomial $f = x^3 - 2 \in \mathbf{Q}[x]$.
   a) Determine the degree $[K : \mathbf{Q}]$.
   b) Determine the Galois group of the polynomial f.
   c) Determine all subfields of K.

6. (15pts) A module M over a ring R is said to be *uniform* if for all nonzero submodules B, C of M, $B \cap C \neq \{0\}$.
   a) Give an example of a ring R and a uniform R-module M.
   b) Prove that A is a uniform R-module if and only if for any two nonzero elements $b, c \in A$ there exist $r, s \in R$ such that $rb = sc \neq 0$.
   c) Prove that if A is a uniform R-module, and if U and V are simple R-submodules of A, then $U = V$. (Recall that a nonzero module is simple provided it contains no submodules other than itself and the zero submodule.

7. (10pts) Let K be a field of characteristic p and let $f(x) = x^p - x - c \in K[x]$.
   a) Show if α is a root of $f(x)$, so is $\alpha + 1$.
   b) Show that $K(\alpha)$ is the splitting field of $f(x)$.

8. (10pts) Let G be a group of odd order. Prove that if $a \in G$ then there exists a *unique* $g \in G$ such that $g^2 = a$.