

Algebra Qualifying Examination
May, 2000

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes:

1. All rings are unitary. All modules are unitary.
 2. \mathbb{Q} is the rationals, \mathbb{R} the reals, \mathbb{C} the complexes, and \mathbb{Z} the integers.
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1. (15 points) Let K be the splitting field of the polynomial $f(x) = x^3 - 5 \in \mathbb{Q}[x]$.
 - a) Determine the degree $[K : \mathbb{Q}]$.
 - b) Determine the Galois group of the polynomial f .
 - c) Determine all subfields of K .
 2. (15 points) Let V be a finite dimensional vector space over \mathbb{R} . A *norm* on V is a function $\|\cdot\|: V \rightarrow \mathbb{R}$ satisfying
$$\begin{aligned}\|\mathbf{u}\| &> 0 \text{ for every nonzero } \mathbf{u} \in V, \\ \|r\mathbf{u}\| &= |r| \|\mathbf{u}\| \text{ for every } r \in \mathbb{R}, \mathbf{u} \in V, \\ \|\mathbf{u} + \mathbf{v}\| &\leq \|\mathbf{u}\| + \|\mathbf{v}\| \text{ for every } \mathbf{u}, \mathbf{v} \in V.\end{aligned}$$
A linear transformation $L: V \rightarrow V$ is an *isometry* for $\|\cdot\|$ on V if $\|L(\mathbf{u})\| = \|\mathbf{u}\|$ for all $u \in V$.
 - (i) Prove that the set G of isometries for $\|\cdot\|$ on V is a group under function composition.
 - (ii) Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be (finite dimensional) normed real vector spaces with isometry groups G_1 and G_2 respectively. Let $L: V_1 \rightarrow V_2$ be a vector space isomorphism such that $\|L(\mathbf{u})\|_2 = \|\mathbf{u}\|_1$ for all $\mathbf{u} \in V_1$. Prove that G_1 and G_2 are isomorphic.
 - (iii) Let $V = \mathbb{R}$ the one dimensional vector space over \mathbb{R} . Let $\|\cdot\|$ be $\|r\| = |r|$, the absolute value of r . What group is the isometry group for $\|\cdot\|$ on V ?

3. (10 points) Let R be an integral domain. Let M be an R -module. The *rank* of M is the maximal number of R -linearly independent elements of M . Suppose that M has rank n and that x_1, x_2, \dots, x_n is any maximal set of R -linearly independent elements of M . Let $N = Rx_1 + Rx_2 + \dots + Rx_n$ be the submodule generated by x_1, x_2, \dots, x_n .
- Prove that N is isomorphic to the free R -module R^n .
 - Prove that the quotient module M/N is a torsion module.
4. (15 points) Let p be an odd prime. Let G be a group of order p^3 that is *not* abelian. Let C be the center of G .
- Prove that $G/C \cong Z_p \oplus Z_p$.
 - Prove that the map $f: G \rightarrow G$ defined by $f(x) = x^p$ is a group homomorphism. (Suggestion: By (a), for any $x, y \in G$ there is a $z \in C$ such that $yx = xyz$.)
 - Prove that $f(G) \subseteq C$.
5. (10 points) Let p be a prime. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial such that p divides $f(n)$ for all $n \in \mathbb{Z}$. Prove that there exist polynomials $g(x), h(x) \in \mathbb{Z}[x]$ such that $f(x) = pg(x) + (x^p - x)h(x)$.
6. (10 points) Find all groups of order $7 \cdot 11^3$ that have a cyclic subgroup of order 11^3 .
7. (15 points) Let $R = \mathbb{Z} + x\mathbb{Q}[x]$ be the ring of polynomials with rational coefficients whose constant term is an integer. R is an integral domain.
- What are the units of R ?
 - Show that the polynomial x is not an irreducible element of R .
 - Let (x) be the ideal of R generated by x . Show that $R/(x)$ is not an integral domain.
 - Is R a unique factorization domain? Justify your answer.
8. (10 points) Let V be a finite dimensional vector space over the field F . Let $L: V \rightarrow V$ be a linear transformation. Using L , the vector space V is an $F[x]$ -module by defining $f(x) \cdot \mathbf{v} = f(L)\mathbf{v}$ for all $f(x) \in F[x], \mathbf{v} \in V$
- Without citing any theorems, prove that there is a nonzero polynomial $g(x) \in F[x]$ such that $g(L) = 0$.
 - Suppose $f(L) = 0$ and suppose $f(x) = f_1(x)f_2(x)$ where $f_1(x)$ and $f_2(x)$ are relatively prime. Let $W_1 = \ker f_1(L), W_2 = \ker f_2(L)$. Prove that $V = W_1 \oplus W_2$ (as $F[x]$ -modules).