Algebra Qualifying Examination

May, 2001

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes: All rings have an identity. All modules are unitary. **Q** denotes the rationals, **R** the reals, **C** the complexes, and **Z** the integers.

1. (8 points) Let $T : V \to V$ be a linear transformation on a vector space V of dimension 7, whose minimal polynomial is x^2. What are the possible values of the dimension of the kernel of T? Justify your answer.

2. (27 points) Suppose G is a finite group. Let p be a prime. Suppose H is a normal p-subgroup of G. Prove the following.

   a. H is contained in each Sylow p-subgroup of G.

   b. If K is a normal p-subgroup of G, then HK is a normal p-subgroup of G.

   c. Let $O_p(G)$ be the subgroup generated by all normal p-subgroups of G. Show that $O_p(G)$ is equal to the intersection of the Sylow p-subgroups of G.

3. (8 points) Let $A_1, A_2, ..., A_n$ be left ideals of a ring R with identity such that $R = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ as abelian groups. Prove that for $i = 1, 2, ..., n$ there exist $u_i \in A_i$ such that, for all $a_i \in A_i$, $a_i u_i = a_i$ and $a_i u_j = 0$ for $i \neq j$.

4. (16 points) Recall that $\mathbf{Z}[i] = \{a + bi : a, b \in \mathbf{Z},\ i^2 = -1\}$ is the ring of Gaussian integers. Let $R = \mathbf{Z}[i]/I$ where $I = (1 + 3i)$.

   a. Prove that $(a + bi) + I = (a + 3b) + I$, for all $a, b \in \mathbf{Z}$.

   b. Prove that R is isomorphic to $\mathbf{Z}/10\mathbf{Z}$.

5. (8 points) Let $A,\ B,\ C$ be $R-$modules, where R is a commutative ring.
Show that there is an $R-$module homomorphism
$\phi : \mathrm{Hom}_R(B, C) \otimes_R \mathrm{Hom}_R(A, B) \to \mathrm{Hom}_R(A, C)$ such that $\phi(t \otimes s) = t \circ s$, where $t \in \mathrm{Hom}_R(B, C)$ and $s \in \mathrm{Hom}_R(A, B)$.

6. Let K be a field of characteristic p, and let $f(x) = x^p - x - 1 \in K[x]$.

   a. (4 points) Show that, if α is a root of $f(x)$, then so is $\alpha + 1$.

   b. (6 points) Show that $K(\alpha)$ is a Galois extension of K.

   c. (6 points) What are the possibilities for the Galois group of $f(x)$? Justify your answer.

7. (9 points) Let $f(x) = x^4 + bx^2 + c \in \mathbf{Q}[x]$. Let F be the splitting field (in $\mathbf{C}$) of $f(x)$. Prove that the dimension of F over $\mathbf{Q}$ is $1, 2, 4$ or 8.

8. (8 points) Let t be an indeterminate. Find $[\mathbf{Q}(t) : \mathbf{Q}(t^4)]$ and $\mathrm{Aut}_{\mathbf{Q}(\mathbf{t^4})}\mathbf{Q}(t)$.
Your work should justify your answer.