Algebra Qualifying Examination

May 2002

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes:

Each ring R referred to in this exam will be assumed to contain a multiplicative identify 1_R. Every left R-module M will have the property that $1_R \cdot m = m$ for all $m \in M$, and every homomorphism of rings $\varphi\colon\ R \to S$ will be assumed to satisfy the property that $\varphi(1_R) = 1_S$. $\mathbb{Q}$ is the field of rational numbers, $\mathbb{R}$ the field of real numbers, $\mathbb{C}$ the field of complex numbers, $\mathbb{Z}$ the ring of integers, and $\mathbb{F}_q$ a finite field with q elements.

1. (14pts) Classify groups of order 20 (abelian and non-abelian). Express each group up to isomorphism in terms of generators and relations and indicate the number of elements of each order.
2. (12pts) Let G be a finite group acting on itself by conjugation.
   a) Explain the class equation for this action, i.e. write the formula and define the various terms.
   b) Find the various conjugacy classes when $G = S_4$ and verify the formula.
3. (12pts) Let R be a commutative ring with $1_R \neq 0_R$.
   a) Show R contains a minimal prime ideal P, i.e. a prime ideal which contains no smaller prime ideal.
   b) Assume R has a finite number of minimal primes $P_1, \ldots, P_n$. Show that $\bigcup_{i=1}^{n} P_i$ consists of zero divisors. (Hint: you may use the fact that the intersection of all prime ideals in R is the set of nilpotent elements.)
4. (14pts) Let R be a commutative ring with $1_R \neq 0_R$ and S a multiplicative set (we assume $1_R \in S$).
   a) Give the universal property that characterizes the localization $S^{-1}R$ and explain how to construct $S^{-1}R$.
   b) What is the kernel of the canonical homomorphism $\varphi\colon\ R \longrightarrow S^{-1}R$.

c) Let I, J be distinct prime ideals in R with $I \cap S = \emptyset$ and $J \cap S = \emptyset$. Show that the ideals generated by $\varphi(I)$ and $\varphi(J)$ are prime in $S^{-1}R$ and distinct.

5. (12pts) Let $x^4 + ax^2 + b \in \mathbb{Q}[x]$, $a, b \in \mathbb{Q}$, be an irreducible polynomial with Galois group G. What are the possible Galois groups and give conditions on a and b for each to occur.

6. (12pts) Find the Galois group of the splitting field of $x^3 - 4x + 2$ over $\mathbb{Q}$. Determine explicitly the intermediate fields (if any) which are Galois over $\mathbb{Q}$.

7. (12pts) Show that a finitely generated projective module P over a commutative local ring R is a free module. Hint: Consider free modules F with a surjection $\pi\colon\ F \to P$. Pick one with the fewest elements in a basis. Show $\ker \pi \subset mF$ where m is the unique maximal ideal of R. Then use the fact that P is projective and Nakayama's Lemma: If N is a finitely generated module over a local ring R and $mN = N$ where m is the unique maximal ideal of R then $N = 0$.

8. (12pts) Let F be a finite Galois extension of the field K and consider map

$$\begin{array}{ccc} F \times F & \xrightarrow{\varphi} & K \\ (x, y) & \longmapsto & Tr_K^F(xy) \end{array}$$

where Tr_K^F is the usual trace.

a) Show that φ is K-bilinear.

b) In the case where $F = \mathbb{F}_9$ and $K = \mathbb{F}_3$ represent φ by an explicit 2×2 matrix.