Algebra Qualifying Examination
May, 2003

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes:

1. All rings are unitary. All modules are unitary.
2. $\mathbb{Q}$ is the rationals, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, and $\mathbb{Z}$ the integers.

1. (21 points) Let Q be the group of quaternions, a group of order 8.
   (i) Show that Q is not isomorphic to a subgroup of S_4.
   (ii) Use Sylow theory to show that Q is not isomorphic to a subgroup of S_5.
   (iii) Find an n such that Q is isomorphic to a subgroup of S_n and give the isomorphism.
2. (14 points) Let R be a ring. Recall that a short exact sequence
$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$
   of R-modules and R-homomorphisms is split if there exists an R-homomorphism $g'\colon\ C \to B$ such that $gg' = 1_C$.
   Prove that the short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is split if and only if there exists an R-homomorphism $f'\colon\ B \to A$ such that $f'f = 1_A$.
3. (7 points) Prove that every finite field can be obtained as a homomorphic image of a single integral domain D where $D \subseteq \mathbb{R}$.
   (Suggestion: Let $c \in \mathbb{R}$ be transcendental over $\mathbb{Q}$ and consider $Z[c]$.)
4. (7 points) Let D be a division ring. Let K be the center of D. Then K is a field and D is a vector space over K. Prove that if the dimension of D over K is finite and K is algebraically closed, then $D = K$.

5. (15 points) Let $G \neq (1)$ be a possibly infinite group whose subgroups are linearly ordered by inclusion. In other words, if H and K are subgroups of G, then either $H \subseteq K$ or $K \subseteq H$.

   a. Prove that G is an abelian group and that the orders of the elements of G are all powers of the same prime p.

   b. If $G_n = \{g \in G \mid g^{p^n} = 1\}$, prove that $|G_n| \leq p^n$.

6. (22 points) Let E be a finite degree field extension of the rationals $\mathbb{Q}$ and suppose that $f(x)$ is a monic irreducible polynomial in $E[x]$.

   i. Show that there exists a unique monic irreducible polynomial $g(x) \in \mathbb{Q}[x]$ such that $f(x)$ divides $g(x)$ in $E[x]$.

   ii. Let $g(x)$ be as above. If E is a splitting field over $\mathbb{Q}$ for some polynomial in $\mathbb{Q}[x]$, show that the degree of $f(x)$ divides the degree of $g(x)$.
   Suggestion: Let $\mathrm{Gal}(E/\mathbb{Q}) = \{\sigma_1, \ldots, \sigma_k\}$ and consider $\sigma_1(f(x))\sigma_2(f(x)) \cdots \sigma_k(f(x))$.

   iii. Give an example to show that the degree of $f(x)$ need not divide the degree of $g(x)$ in general.

7. (14 points) Let G be the multiplicative group of all 2×2 matrices over the integers $\mathbb{Z}$ whose determinant is equal to 1. Notice that G acts by left multiplication on the set Ω of all 1-dimensional subspaces of the 2-dimensional column space $\mathbb{Q}^2 = \left\{ \binom{r}{s} \mid r, s \in \mathbb{Q} \right\}$ over the rational numbers $\mathbb{Q}$.

   i. Find all elements of G which act trivially, that is which fix every element of Ω.

   ii. Prove that G acts transitively. In other words, show that Ω is an orbit under the action of G.