Algebra Qualifying Examination
May, 2004

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes:

1. All rings are unitary. All modules are unitary.
2. $\mathbb{Q}$ is the rationals, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, and $\mathbb{Z}$ the integers.

Questions

1. Prove that every homomorphic image of a divisible group is divisible. Deduce that neither $\mathbb{Q}$ nor $\mathbb{Q}/\mathbb{Z}$ have proper subgroups of finite index.

2. Let G be a finite group acting transitively on the set A, with $|A| > 1$. Show that

$$|\cup_{a\in A} G_a| < |G|,$$

where, for $a \in A$, G_a denotes the stabilizer of a in G . Deduce that there exists some $\sigma \in G$ such that $\sigma(a) \neq a$, for all $a \in A$.

3. A ring R is called a Boolean ring if $a^2 = a$, for all $a \in R$.
   (a) Prove that $a = -a$ holds for all a in a Boolean ring.
   (b) Prove that every Boolean ring is commutative.
   (c) Prove that the only Boolean ring that is an integral domain is $\mathbb{Z}/2\mathbb{Z}$.

4. Let $G = \{g_1, \dots, g_n\}$ be a finite group and R be a ring.
   (a) Prove that the element $C = g_1 + g_2 + \cdots + g_n$ is in the center of the group ring $R[G]$.
   (b) Prove that, if the characteristic of the ring R divides n, then $C^2 = 0$.

5. Prove that the ring of all rational numbers with odd denominators is a local ring whose unique maximal ideal is the principal ideal generated by 2.

6. Prove that the rings $F[x,y]/(y^2 - x)$ and $F[x,y]/(y^2 - x^2)$ are not isomorphic for any field F.

7. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $u \otimes w$ for any $u, w \in \mathbb{R}^2$.

8. Prove that the following are equivalent for a ring R :

   (i) Every R-module is projective.

   (ii) Every R-module is injective.

9. (a) Let A be a 3×3 matrix with entries in $\mathbb{C}$ and let $x^3 - x$ be the characteristic polynomial of A. Prove that A can be diagonalized.

   (b) Let A be an $n \times n$ matrix with entries in $\mathbb{C}$ and let $A^3 = A$. Prove that A can be diagonalized.

   Are the statements in (a) and (b) true over any field F?

10. Let K be a normal extension of $\mathbb{Q}$ of degree n. Let $G(K/\mathbb{Q}) = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ be the Galois group of K over $\mathbb{Q}$.

    (a) For $c \in K$ let $f(x) = (x - \sigma_1(c))(x - \sigma_2(c)) \ldots (x - \sigma_n(c))$. Prove that $f(x) \in \mathbb{Q}[x]$.

    (b) Let $c \in K$. Prove that $K = \mathbb{Q}(c)$ if and only if $\sigma_1(c), \sigma_2(c), \ldots, \sigma_n(c)$ are all different.