

Algebra Qualifying Examination

May, 2005

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes: All rings are unitary. All modules are unitary. \mathbf{Q} is the rationals, \mathbf{R} the reals, \mathbf{C} the complexes, \mathbf{Z} the integers, and \mathbf{Z}_n is the integers modulo n (or $\mathbf{Z}/n\mathbf{Z}$).

1. (10 points) Let V be a finite dimensional vector space over a field F . Let $T : V \rightarrow V$ be a linear transformation of rank 1. Prove that T is either diagonalizable or nilpotent but not both.
2. (10 points) Let $G = \{a_1, a_2, \dots, a_n\}$ be an abelian group of order n . What are the possibilities for $a_1 + a_2 + \dots + a_n$ and when do each occur? Your work should demonstrate that your answer is correct.
3. (10 points) Let $G = S_n$, the symmetric group on n letters. Then G acts on the set $X = \{1, 2, \dots, n\}$ in a natural way, namely, $\sigma \cdot i = \sigma(i)$. For $\sigma \in S_n$ let $\text{Fix}(\sigma) = |\{i \in X : \sigma(i) = i\}|$. So $\text{Fix}(\sigma)$ is the number of elements of X fixed by σ . Determine the integer

$$\sum_{\sigma \in S_n} \text{Fix}(\sigma)$$

and prove your answer.

(over)

4. (20 points) Let $R = \mathbf{Z}[i]$, the ring of Gaussian integers.
- Let $p \in \mathbf{Z}$ be a prime integer. Show p is prime in R if $x^2 + y^2 = p$ has no integral solutions. (Hint: Remember the norm function $N : R \rightarrow \mathbf{Z}$.)
 - Show that 11 does not divide $4n^2 + 1$ for all $n \in \mathbf{Z}$.
5. (20 points) Let R be a commutative ring.
- Prove that, for any R -module M , $M \otimes_R R \cong M$.
 - Compute $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}_2$. Your work should prove that your answer is correct.
6. (10 points) Let $f(x) = x^4 + 1 \in \mathbf{Z}[x]$.
- Show that, for all odd primes p , $f(x)$ divides $x^{p^2} - x$ in $\mathbf{Z}[x]$.
 - Show that $f(x)$ is reducible when considered as a polynomial in $\mathbf{Z}_p[x]$ for each prime p .
7. (5 points) Suppose F is an extension field of a field K . Suppose $a \in F$ is algebraic such that $[K[a] : K]$ is odd. Show that $K(a) = K(a^2) \subseteq F$.
8. (15 points) Let x be an indeterminate. Your work on the following should demonstrate that your answer is correct.
- Find $[\mathbf{Q}(x) : \mathbf{Q}(x^4)]$.
 - Find $|\text{Aut}_{\mathbf{Q}(x^4)} \mathbf{Q}(x)|$.
 - Show whether or not $\mathbf{Q}(x)$ is Galois over $\mathbf{Q}(x^4)$.