Algebra Qualifying Examination
May, 2006

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

Notes:

1. All rings are unitary. All modules are unitary.
2. $\mathbb{Q}$ is the rationals, $\mathbb{R}$ the reals, $\mathbb{C}$ the complexes, and $\mathbb{Z}$ the integers.

Problems

1. (10 points) Let A and B be ideals of a ring R. Prove that $A/A \cap B$ and $(A+B)/B$ are isomorphic as rings. (Here, the factor rings may not have identity elements.)
2. (10 points) Prove that, up to isomorphism, S_3 is the only group of order 6 that is not abelian.
3. (20 points, 4 each) This is a counting problem. For the sake of partial credit, explain your reasoning.
   (i) How many nonisomorphic abelian groups of order 72 are there?
   (ii) How many elements in S_7 commute with the permutation $(1\ 3)(4\ 5\ 6)$?
   (iii) How many subfields of the field $K = \mathbb{Q}(i, \sqrt{2})$ are there (including K itself)?
   (iv) How many subfields of the finite field $K = GF(p^{12})$ are there (including K itself), where p is a prime?
   (v) How many Sylow 3-subgroups are there in S_6?
4. (15 points) Let K be a finite field extension of the field $F, [K : F] = n$. Let u be an element in K. Define $L_u\colon\ K \to K$ by $L_u(a) = ua$. L_u is an F-linear transformation on the n-dimensional vector space K over F. Let $f(x) \in F[x]$ be the characteristic polynomial of L_u.
   (i) Explain why $f(u) = 0$, i.e. show that u is a root of the characteristic polynomial for L_u.
   (ii) Let $g(x) \in F[x]$ be the minimal polynomial for u over F. Show that $g(L_u) = 0$.

(iii) Show that $f(x) = g(x)^m$, where $m = [K : F(u)]$.

5. (15 points) Let V be a vector space of dimension n over the field F. Let $T\colon\ V \to V$ be a linear transformation. Make V into an $F[x]$-module by $f(x) \cdot v = f(T)(v)$ for all $f(x) \in F[x]$ and $v \in V$.

   (i) Show that V is a torsion module, i.e. show that for every $v \in V$ there is a nonzero $f(x) \in F[x]$ such that $f(x) \cdot v = 0$.

   (ii) Suppose W_1 and W_2 are nonzero submodules of V such that V is a direct sum of W_1 and W_2. Show that the characteristic polynomial for T is not irreducible.

   (iii) Let $m(x) \in F[x]$ be the minimal polynomial for T. Suppose $m(x) = f(x)g(x)$ is a proper factorization of $m(x)$ with $(f(x), g(x)) = 1$. Show that V is a module direct sum of two proper submodules.

6. (10 points) Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial. Suppose K is an extension field of $\mathbb{Q}$ that contains a root u of $f(x)$ such that $f(u^2) = 0$. (So u^2 is also a root of $f(x)$.) Prove that $f(x)$ splits in $K[x]$.

7. (10 points) Let H be the subgroup of $G = Z^3$ generated by $\{(1, 2, 4), (-2, 4, 10), (3, -2, 6)\}$. Characterize the factor group G/H according to the fundamental theorem of finitely generated abelian groups.

8. (10 points) Let R be a commutative ring (with 1). Suppose every R-module is free. Prove that R is a field.