Algebra Qualifier Exam
May 22, 2007
1:00-5:00

Instructions:

- There are seven Algebra questions worth a total of 100 points. Individual point values are listed next to each problem.
- Credit awarded for your answers will be based on the correctness of your answer as well as the clarity and main steps of your reasoning. "Rough working" will not be accepted: answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but you will not earn points for using it as a step in your reasoning).

Notation: Throughout, $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ denotes the rational numbers, $\mathbb{R}$ denotes the real numbers, and $\mathbb{C}$ denotes the complex numbers. For an integer n, $\mathbb{Z}_n$ denotes the ring of integers modulo n.

Problems:

1. [10 points] Let G be a group of order 55. Suppose $a \in G$ has order 5 and $b \in G$ has order 11.
   (a) Show that $G = \{a^i b^j \mid 0 \leq i \leq 4,\ 0 \leq j \leq 10\}$.
   (b) Show that $ba = ab^k$ for some k, $1 \leq k \leq 10$.
   (c) Up to isomorphism, how many different possibilities for G are there?

2. [10 points] Let p be a prime such that $p \equiv 3 \pmod 4$. Consider the following rings:
$$\mathbb{Z}_{p^2}, \quad \mathbb{Z}_p \oplus \mathbb{Z}_p, \quad \mathbb{Z}_p[x]/(x^2+1).$$
   (a) Which of these rings are fields?
   (b) Which pairs of these rings are isomorphic?
   (c) Which of these rings are $\mathbb{Z}_p$-modules? Which are $\mathbb{Z}_{p^2}$-modules?
   (d) Which of these rings are free $\mathbb{Z}_p$-modules? Which are free $\mathbb{Z}_{p^2}$-modules?

3. [15 points] Let P be a p-subgroup of a finite group G. Consider the action of P on G/P by
$$g \cdot xP := (gx)P.$$
   (a) Show that $xP \in (G/P)^P \iff x \in N_G(P)$.
   (b) Deduce that $|N_G(P)/P| \equiv |G/P| \pmod p$.
   (c) Now suppose that G is a finite p-group and P is a proper subgroup of G. Show that P is a proper subgroup of $N_G(P)$.

4. [15 points] Let $O(x)$ denote the order of an element x of a group G.

   (a) Let x, y be elements of finite order of a group G such that
   $$\gcd(O(x), O(y)) = 1.$$
   i. Show that $\langle x \rangle \cap \langle y \rangle = \{1\}$.
   ii. Suppose in addition that $xy = yx$. Show that $O(xy) = O(x)O(y)$.

   (b) Consider the following elements of $\mathrm{GL}_2(\mathbb{Q})$:
   $$a := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad b := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$
   i. Show that $O(a) = 4$ and $O(b) = 3$.
   ii. Show that $O(ab) = \infty$.
   iii. Compare with part (a).

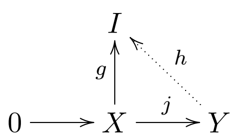
5. [15 points] Give examples of the following objects. Be sure to verify that your examples satisfy the desired properties.

   (a) A non-zero commutative ring with 1 in which $\{0\}$ is not a prime ideal.
   (b) A non-zero ideal in a commutative ring with 1 that is prime but not maximal.
   (c) A unique factorization domain that is not a principal ideal domain.
   (d) An irreducible polynomial over $\mathbb{Q}$ that is Eisenstein for $p = 3$.
   (e) A polynomial over a field that is irreducible but not separable.

6. [20 points] Let $f(x) = x^5 - 3 \in \mathbb{Q}[x]$.

   (a) Let E be the splitting field of f over $\mathbb{Q}$. Describe E explicitly. What is $[E : \mathbb{Q}]$?
   (b) Show that $\mathrm{Gal}(E/\mathbb{Q})$ is isomorphic to a subgroup of $\mathrm{GL}_2(\mathbb{Z}_5)$ consisting of matrices of the form $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$.
   (c) Describe explicitly the action of $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ on your chosen generators for E over $\mathbb{Q}$.
   (d) How many intermediate fields of degrees 4 and 5 over $\mathbb{Q}$ are there between $\mathbb{Q}$ and E?

7. [15 points] Let R be a ring with 1. For this problem, "R-module" will connote "left R-module." Recall that an R-module I is injective if for any injective homomorphism of R-modules $j : X \to Y$ and any R-module homomorphism $g : X \to I$ there exists an R-module homomorphism $h : Y \to I$ so that $h \circ j = g$. That is, there exists an h so that the following diagram commutes:



   (and the bottom row is exact).

   (a) Show that if I is an injective R-module then every short exact sequence of R-modules $0 \to I \to B \to C \to 0$ splits.
   (b) Let R be an integral domain. Let I be an injective R-module. Show that, for every non-zero $a \in R$, the natural map $\phi_a : I \to I$ given by $\phi_a(u) = au$ is surjective.