Algebra Qualifying Examination
May 1998

Directions:

1. Answer all questions. (Total possible is 100 points.)
2. Start each question on a new sheet of paper.
3. Write only on one side of each sheet of paper.

Policy on Misprints:

The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. (15 pts) Let G be a group of order 12 and let H be a Sylow 2-subgroup of G and K a Sylow 3-subgroup of G. Suppose H is normal but K is not. Show G is isomorphic to the alternating group A_4 by considering the action of G on the set consisting of K and its conjugates.

2. (12 pts) The group of symmetries of the cube has order 24.
   a) Determine the number of elements of each possible order.
   b) Determine the number of Sylow 3-subgroups and the number of Sylow 2-subgroups.
   c) Does G have a subgroup isomorphic to A_4? Why or why not?

3. (12 pts) Determine the splitting field of $x^4 + x + 1$ over $\mathbf{F}_2$.

4. (12 pts)
   a) State the Fundamental Theorem of Galois Theory.
   b) Let $\zeta = e^{2\pi i/7}$ be a primitive 7^{th} root of unity in $\mathbf{C}$. Find explicity all intermediate fields of $\mathbf{Q}(\zeta)$ over $\mathbf{Q}$.

5. (13 pts) Let R be a non-zero commutative ring (with $1_R \in R$) and let S be a multiplicative set. Assume $0_R \notin S$. Show there exists a prime ideal P in R such that $S \cap P = \emptyset$.

6. (12 pts) Let A be the group of all ring automorphisms of the polynomial ring $\mathbf{Q}[x]$, where $\mathbf{Q}$ is the field of rational numbers. Characterize the finite subgroups of A.

7. (12 pts) An R-module M is called faithful if $rM = 0$ for $r \in R$ implies $r = 0$. Let M be a finitely generated faithful R-module and let I be an ideal of R such that $IM = M$. Prove $I = R$.

8. (12 pts) Let X and Y be $n \times n$ complex matrices for which $XY = YX$. Prove that X and Y are simultaneously upper triangularizable (i.e. prove there exists an invertible $n \times n$ matrix P such that PXP^{-1} and PYP^{-1} are upper triangular.