Algebra Qualifying Exam May 1999

Directions:

(1) Answer all questions. (Total possible is 100 points.)

(2) Start each question on a new sheet of paper.

(3) Write only on one side of each sheet of paper.

Notes: The rational numbers will be denoted by $\mathbf{Q}$, the integers by $\mathbf{Z}$, the non-negative integers (including 0) by $\mathbf{N}$ and the positive integers by $\mathbf{N}^*$.

1. (15 points)

Let G be a finite group. An automorphism α of G is called *fixed point free* if $\alpha(x) = x$ implies $x = e$, the identity element of G. Let α be a fixed point free automorphism of G of order n.

(i) If $\gcd(i, n) = 1$ prove that α^i is also fixed point free.

(ii) Prove that the mapping $\mu : x \mapsto x^{-1}\alpha(x)$ is a permutation of G.

(iii) Prove that the elements x and $\alpha(x)$ are conjugate if and only if $x = e$. (Hint: Use (ii))

2. (15 points)

(i) Prove that every group of order p^2 (p prime) is abelian.

(ii) Prove that no group of order 56 is simple.

3. (15 points)

Let R be a commutative ring with identity and let $f \in R$ be a non-nilpotent element. Consider the multiplicative set $S = \{1_R, f, f^2, f^3, \ldots\}$ and the localization $S^{-1}R$, also denoted R_f, and the canonical homomorphism:

$$\phi : R \to R_f \quad \text{such that} \quad a \mapsto a/1_R$$

(i) Show that there is a natural 1-1 correspondence between the prime ideals in R_f and the prime ideals in R which do not contain f.

(ii) Give an example to show where this correspondence fails when we drop the "prime" assumption.

4. (10 points)

Let I be the ideal of all polynomials $f(x) \in \mathbf{Z}[x]$ with $f(0) = 0$ and let J be the ideal of all polynomials in $\mathbf{Z}[x]$ such that $f(0)$ is an even integer. Show that

(i) I is a prime ideal.

(ii) J is a maximal ideal.

(iii) I is a principal ideal.

(iv) J is NOT a principal ideal.

5. (15 points)

For both parts of this question, A is a non-zero commutative ring with identity and M is a unitary A-module.

(i) Prove that M is a finitely generated A-module if and only if M is isomorphic to a quotient of A^n for some $n \in \mathbf{N}^*$.

(ii) Let M be a finitely generated A-module and I an ideal of A contained in the intersection of all maximal ideals of A. Prove that if $IM = M$ then $M = 0$.

6. (15 points)

The following three questions are unrelated to each other.

(i) Let θ be a root of $x^3 - 2x - 2 \in \mathbf{Q}[x]$. Compute $(1+\theta)/(1+\theta+\theta^2)$ in $\mathbf{Q}(\theta)$.

(ii) Determine the minimal polynomial of $1 + i$ over $\mathbf{Q}$, where $i^2 = -1$.

(iii) Give an example of a separable extension of $\mathbf{Q}$ that is not Galois over $\mathbf{Q}$. Justify your answer.

7. (15 points)

(i) Find the Galois group over $\mathbf{Q}$ of $f(x) = x^4 - 4x + 2$.

(ii) Calculate the splitting field K of $x^4 + x^2 - 6$ over $\mathbf{Q}$ and draw the lattice of all fields that are intermediate between $\mathbf{Q}$ and K. Which of these intermediate fields are Galois over $\mathbf{Q}$?