Applied Mathematics Qualifying Exam

January 2001

Do any six of seven problems in this exam. Clearly show all of your work.

Policy on Misprints: The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. Let H be a Hilbert space and $K \subset H$ be a nonempty closed convex set. It is known that for each $u \in H$, there exists a unique $u^* \in K$ such that

$$\|u - u^*\| \leq \|u - w\| \quad \forall w \in K.$$

The solution u^* is called the projection of u onto K and denoted by $P_K(u)$.
(a) Prove that for each $u \in H, u^* = P_K(u)$ if and only if $u^* \in K$ and

$$\langle u^* - u | w - u^* \rangle \geq 0 \quad \forall w \in K;$$

(b) State the Riesz Representation theorem and the Banach fixed point theorem;
(c) Assume that $a : H \times H \to R$ is a bounded and strongly positive bilinear form and $b : H \to R$ is a bounded linear functional. Prove that there exists a unique $u^* \in K$ such that

$$a(u^*, w - u^*) \geq b(w - u^*) \quad \forall w \in K.$$

2. Given $u \in L^2(0, 2\pi)$, define for each integer k the Fourier coefficient

$$u_k = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-\mathbf{i}kx} dx.$$

Let $a = (a_1, a_2, ...) \in \ell^2$ and $f \in L^2(0, 2\pi)$ be given. Define

$$(Au)(x) = \sum_{k=1}^{\infty} \frac{a_k}{k} \cos(|u_k|^2) e^{\mathbf{i}kx}, \quad x \in [0, 2\pi].$$

Prove that the problem $u - Au = f$ has at least one solution $u \in L^2(0, 2\pi)$.

3. Let G be a bounded open subset of R^n and $H_0^1(G)$ denote the usual Sobolev space obtained by completing the set of functions $C_0^\infty(G)$ (infinitely smooth functions with compact support in G) with respect to the norm

$$\|f\|_{H^1(G)}^2 = \int_G \left(f^2(x) + |\nabla f(x)|^2\right) dx$$

where ∇f denotes the distribution gradient of $f \in H^1(G)$.

(a) State the Poincare-Friedrichs inequality theorem;

(b) Let A be a positive definite, symmetric $n \times n$ matrix. Prove that the bilinear form

$$a(u,v) = \int_G (A\nabla u) \cdot \nabla v \, dx$$

is bounded and strongly positive on the Sobolev space $H_0^1(G)$.

4. For $s \geq 0$, let $h^s = \{(u_n)_{n=-\infty}^{n=+\infty} \mid \sum_{n=-\infty}^{n=+\infty} |n|^{2s}|u_n|^2 < \infty\}$. It is known that h^s is a Hilbert space and that if $s \geq 0$, then h^s is included in l^2 with a continuous injection.

Let $H^s = \{f, \ f(x) = \sum_{n=-\infty}^{n=\infty} u_n \mathrm{e}^{\,\mathbf{i}2n\pi x}, \text{a.e.} x \in [0,1], \text{with } (u_n) \in h^s\}$.

(a) Prove that if $f \in H^s, \ s \geq 0$, then $f \in L^2(0,1)$;

(b) Prove that if $f \in C^1[0,1]$ with $f(0) = f(1)$ then $f \in H^1$;

(c) Prove that the injection of H^1 in $C^0[0,1]$ is compact (you may assume that the set of C^1-functions is dense in H^1.)

5. Recall that the Sobolev space $H^s(R^n)$ can be defined in terms of the Fourier transform as the completion of $C_0^\infty(R^n)$ with respect to the norm

$$\|u\|_{H^s(R^n)}^2 = \int |\hat{u}(\xi)|^2(1+|\xi|^2)^s \, d\xi.$$

Define the *trace operator* $T : C_0^\infty(R^n) \to C_0^\infty(R^{n-1})$ by

$$(Tu)(x') = u(0,x'),$$

where $x \in R^n$ is denoted $x = (x_1, x')$.

Prove that T extends to a bounded linear operator

$$T : H^s(R^n) \to H^{s-1/2}(R^{n-1}), \quad \text{if } s > 1/2.$$

You may use without proof the fact that for $s > 1/2$ there exists a constant C such that

$$\int (1+|\xi'|^2+\xi_1^2)^{-s} \, d\xi_1 \leq C(1+|\xi'|^2)^{-s+1/2},$$

where $\xi = (\xi_1, \xi')$.

6. Let $C_0[0,1]$ denote the set of all continuous real-valued functions f on the interval $[0,1]$ such that $f(0) = f(1) = 0$. Let $J : C_0[0,1] \to R$ be defined by

$$J(f) = \int_0^1 (f(x) - x)^2 \, dx$$

(a) Prove or disprove the existence of a function $f_0 \in C_0[0,1]$ such that $J(f_0) \leq J(f)$ for all $f \in C_0[0,1]$.

(b) Given $C > 0$, let $\mathcal{B} = \{f \in C_0[0,1] : |f(x) - f(y)| \leq C|x - y| \text{ for all } x, y \in [0,1]\}$. Prove or disprove the existence of a function $f_1 \in \mathcal{B}$ such that $J(f_1) \leq J(f)$ for all $f \in \mathcal{B}$.

7. Let $a \in C^1[0,1]$ be given, with $a(x) \geq a_0 > 0$. Consider the problem

$$\begin{cases} -(a(x)u'(x))' = \lambda u(x), & x \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

(a) Prove that there exists an infinite sequence of distinct pairs (λ_k, u_k), (where $\lambda_k > 0$ is a scalar, and $u_k \neq 0$ is a function) which satisfies the above equation.

(b) Prove that if $\lambda_j \neq \lambda_k$ then u_j and u_k are orthogonal in an appropriate sense.