Applied Mathematics Qualifying Exam

January 2002

Do any 6 of the 7 problems in this exam. Clearly show all of your work.

Policy on Misprints. The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. a. State the Banach fixed point Theorem (also called the Contraction mapping principle).

   b. Show that if the nth power of an operator, namely T^n for some $n > 1$, satisfies the conditions of the Banach fixed point theorem, then the operator T itself has a unique fixed point.

   c. Consider the initial value problem

   $$x'(t) = f[x(t), t], \quad x(t_0) = a$$

   where a is a constant. Further assume $x(t) \in C[t_0, t_1]$ is a continuous function and that f satisfies the Lipschitz condition

   $$|f[x_1(t), t] - f[x_2(t), t]| \leq M|x_1 - x_2|(t)$$

   for some positive constant M and $t \in [t_0, t_1]$. Show that the initial value problem has a unique solution in $C[t_0, t_1]$ by showing that the operator

   $$T[x](t) := \int_{t_0}^{t} f[x(\tau), \tau] d\tau + a$$

   has a fixed point.

2. a. Define what is meant by test function and distribution.

   b. Let $f(x) = |x|$ be defined on R. Find the *distributional* derivatives f' and f'' of f on R. Show your work.

   c. Find all *distributional* solutions to $\frac{du}{dx} = 0$. Again, the method of proof is more important than the answer here.

3. a. Let X and Y be Banach spaces. Define what it means for an operator $K\colon\ X \to Y$ to be compact.

   b. Prove the following: Let X and Y be Banach spaces and all operators are from X into Y. If a linear operator K can be approximated (in the operator norm) by a sequence of compact linear operators K_n such that $\lim_{n\to\infty} \|K_n - K\| \to 0$, then K is a compact operator.

c. Let $Kf(x) = \int_a^b k(x,y)f(y)dy,\ x \in [a,b]$ $(k(x,y)$ and $f(x)$ are real-valued). Further assume $\int_a^b \int_a^b k^2(x,y)dxdy < \infty$ and that there is a complete set of orthonormal functions on $L^2([a,b]\times[a,b])$ of the form $\{\phi_j(x)\phi_k(y)\}_{j,k=1}^{\infty}$ for which

$$k(x,y) = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{j,k}\phi_j(x)\phi_k(y)$$

Show that $K\colon\ L^2[a,b] \to L^2[a,b]$ is a compact operator.

4. a. Construct the Green's function for the differential operator $L = (d^2/dx^2) + k^2$, where $k^2 > 0$, with the boundary conditions $u(0) = 0$, $u(1) = 0$.

   b. What are the values of k^2 for which the construction is impossible?

5. Let

$$J(x) = \int_{-1}^{0} |x(t)|dx + \int_0^1 |x(t)-1|dt \qquad \forall x \in C[-1,1].$$

   a. Explicitly construct a minimizing sequence $x_n \in C[-1,1]$ such that

$$\lim_{n\to\infty} J(x_n) = \inf_{x\in C[-1,1]} J(x).$$

   b. Prove or disprove that J attains its minimum in $C[-1,1]$.

   c. Determine for which $x \in C[-1,1]$, the Gateaux differential $\delta J(x;h)$ exists and is linear in each $h \in C[-1,1]$.

6. Let Ω be a nonempty, bounded, open set in R^n, $n \geq 1$ and recall

$$W_0^{1,2}(\Omega) := \text{closure (in } W^{1,2}(\Omega) \text{ norm) of } C_0^\infty(\Omega)\cap W^{1/2}(\Omega).$$

Consider the semilinear elliptic partial differential equation

$$(*) \qquad \left\{\begin{matrix} \Delta u(x) + k\sin(u(x)) = 0 \\ u(x) = 0 \text{ on } \partial\Omega, \end{matrix} \quad , x\in\Omega, \quad u \in W_0^{1,2}(\Omega)\right\}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator, k is a parameter and $\partial\Omega$ is the boundary of Ω. Assume that $\int_\Omega \cos(u(x))dx$ is weakly lower semicontinuous in u.

Prove that $(*)$ has at least two nontrivial solutions if $k > \lambda_1$ where λ_1 is the first eigenvalue of $(-\Delta)$.

7. Consider the initial-boundary-value problem

$$(**) \qquad \left\{\begin{array}{ll} u_t(x,t) + cu(x,t) - u_{xx}(x,t) = 0, & 0 < x < 1, \quad t > 0 \\ u_x(0,t) = u_x(1,t) = 0 & t > 0, \\ u(x,0) = u_0(x) & 0 < x < 1 \end{array}\right.$$

for some constant $c > 0$ and $u_0 \in X = L_2(0,1)$. Define the operator B by

$$(Bv)(x) = -v''(x) + cv(x) \text{ with } D(B) = \{v \in C^\infty[0,1]\colon\ v' \in C_0^\infty[0,1]\}.$$

a. Verify that the operator B is linear, symmetric and strongly monotone;

b. Let A be the Friedrichs extension of B with the energetic space X_E. Prove that the embedding

$$X_E \subset X$$

is compact;

c. Prove that $(**)$ has a unique generalized (mild) solution;

d. Write the solution as a Fourier series in terms of (λ_n, u_n), the eigenvalues and eigenfunctions of A.