

**APPLIED ANALYSIS QUALIFYING EXAMINATION
JANUARY 2003**

*Hand in all the problems that you attempt. Your grade will be based
on your best 7 answers.*

Policy on misprints. *The qualifying examination committee tries to proof-read the examinations as carefully as possible. Nevertheless, there may be a few misprints. If you are convinced that a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.*

Q1. State and prove the Babuška–Lax–Milgram theorem.

Q2. (a) Define the Fréchet derivative of a function $F : X \rightarrow Y$, where X and Y are Banach spaces.

(b) Let X be \mathbb{R}^2 with the norm $\|(a, b)\|_1 = |a| + |b|$. For which points $x \in X$ does the derivative $F'(x)$ exist, where $F(x) = \|x\|_1$?

Q3. Let K be a norm closed convex subset of a real Hilbert space H . For each $x \in H$, let P_x denote the closest point in K to x . Prove that

$$\|P_x - P_y\| \leq \|x - y\| \text{ for all } x, y \in H.$$

Q4. Find the Green's function $g(s, t)$ for the differential equation

$$x''(s) - x(s) = y(s), \quad x(0) = x'(0) = 0, \quad 0 \leq s \leq 1,$$

and express the solution of the equation in terms of it. You may assume that $g(s, t)$ has the form

$$g(s, t) = \begin{cases} 0, & 0 \leq s \leq t \leq 1, \\ \alpha(t)u(s) + \beta(t)v(s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Q5. (a) State Euler's equation in the calculus of variations, and state the general problem to which it applies.

(b) On the vertical cylinder $x^2 + y^2 = 1$, $z \geq 0$, find the shortest path from the point $(1, 0, 0)$ to the point $(0, 1, 1)$. You may assume the parameterization $(\cos t, \sin t, z(t))$.

Q6. (a) State the contraction mapping theorem.

(b) Consider the integral equation

$$x(t) + \int_0^t k(s, t)x(s) ds = y(t), \quad x(t) \in C[0, 1],$$

where $y(t)$ is a fixed function in $C[0, 1]$ and $k(s, t)$ is a fixed function in $C[0, 1]^2$. Prove that the equation has a unique solution. (Hint: view the integral term as a linear operator and show that some power of it is a contraction)

Q7. Let P_n be the space of polynomials of degree at most n on $[0, 1]$, let $X = \{x_j\}_{j=1}^k \subseteq [0, 1]$ be a finite set, and let T be the operator which restricts each $p \in P_n$ to X . Further suppose that

$$\frac{1}{2}\|p\|_\infty \leq \|Tp\|_\infty \leq \|p\|_\infty \quad \text{for all } p \in P_n.$$

Now fix $x_0 \in [0, 1]$ and define a linear functional λ on $P_n|_X$ by

$$\lambda(Tp) = p(x_0), \quad p \in P_n.$$

(a) Show that λ is well defined and that $\|\lambda\| \leq 2$.

(b) Show that there exist scalars $\{c_j\}_{j=1}^k$ such that

$$\sum_{j=1}^k |c_j| \leq 2 \quad \text{and} \quad p(x_0) = \sum_{j=1}^k c_j p(x_j) \quad \text{for all } p \in P_n.$$

Q8. (a) Define the Fourier transform \hat{f} for $f \in L^1(\mathbb{R})$.

(b) Suppose that f and \hat{f} are both in $L^1(\mathbb{R})$, and that

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\omega) e^{i\omega x} d\omega, \quad \text{for all } x \in \mathbb{R}.$$

Assume also that

$$\sum_{k \in \mathbb{Z}} \hat{f}(\omega + 2\pi k) = 1, \quad \text{for all } \omega \in \mathbb{R}.$$

Show that, for any $n \in \mathbb{Z}$,

$$f(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

Q9. (a) Define the space \mathcal{S} of test functions, give the definition of a distribution, and also define convergence of a sequence of distributions.

(b) Let $f(x)$ be a non-negative continuous function defined on \mathbb{R} and suppose that

$$\int_{\mathbb{R}} f(x) dx = 1.$$

For $k \in \mathbb{N}$, define

$$f_k(x) = kf(kx), \quad x \in \mathbb{R},$$

and let S_k be the distribution defined by

$$\langle S_k, \phi \rangle = \int_{\mathbb{R}} f_k(x) \phi(x) dx, \quad \phi \in \mathcal{S}.$$

Show that

$$\lim_{k \rightarrow \infty} S_k = \delta_0.$$

(Hint: consider the function $\psi(x) = \phi(x) - \phi(0)$)