Applied Mathematics Qualifying Exam

January 2005

Do any 7 of the 9 problems in this exam. Clearly show all of your work. Please indicate which 2 of the 9 problems you will skip in your work.

Policy on Misprints. The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

1. a. State the Banach fixed point (Contraction mapping) Theorem.

   b. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous and satisfy a Lipschitz condition
   $$|f(s,t_1) - f(s,t_2)| \le \ell |t_1 - t_2|, \quad \forall s \in [a,b],$$
   for some constant $\ell > 0$ which depends only on f. Prove that the initial value problem
   $$x'(t) = f(t, x(t)), \quad x(a) = x_0$$
   where x_0 is a constant, has a unique solution $x \in C[a,b]$.

2. It is known that if H is an inner product space and A is a compact Hermitian linear operator on H, then A has at least one eigenvalue λ such that $|\lambda| = \|A\|$.
   Prove that a compact Hermitian linear operator A on an inner product space H is of the form
   $$Ax = \sum_{k=1}^{\infty} \lambda_k \langle x, e_k \rangle e_k, \quad \forall x \in H,$$
   where $\langle \cdot, \cdot \rangle$ is the inner product, $\{e_k\}$ is some orthonormal sequence in H and $\{\lambda_k\}$ is some real sequence such that $Ae_k = \lambda_k e_k$ and $\lim_{k\to\infty} \lambda_k = 0$.

3. Let $X = C[0,1]$ and $f(x) = \max_{0 \le t \le 1} x(t)$ for $x \in X$.

   a. Discuss the Frechet differentiability of f;

   b. Find $f'(x)$ where it exists.

4. Find the Green's function for the problem
   $$x'' - 3x' + 2x = y, \quad x(0) = x(1) = 0$$
   where y is a given continuous function on $[0,1]$.

5. Let X be an inner product space with inner product $\langle\cdot,\cdot\rangle$, $\{u_1, ..., u_n\}$ and $\{v_1, ..., v_n\}$ be two sets (not necessarily linearly independent) in X. An operator $A : X \to X$ is defined by $Ax = \sum_{i=1}^{n} \langle x, u_i \rangle v_i$. For given $b \in X$, consider solving the problem $\min_{x \in X} \|Ax - b\|$.

   a. Discuss the solvability of the problem;

   b. In solving the problem, discuss the solvability of linear systems involved;

   c. Describe the solution set.

6. a. Given $f : R \to R$, define the Fourier transform of f including what assumptions on f are needed.

   b. Let $g(t) = f(at + b)$, with a,b $\in R$ fixed and $a \neq 0$. Assuming that the Fourier transforms $\hat{f}, \hat{g}$ exist, compute $\hat{g}$ in terms of $\hat{f}$.

   c. Show that if the Fourier transform $\hat{f}$ exists, then it is a uniformly continuous function. You may assume $\hat{f} : R \to R$.

7. Work either a. or b.

   a. Prove this theorem: Let $[P_n]$ be a sequence of projections on a Banach space X, and assume that $P_n y \to y$ for each y in X. Let $b \in X$ and $A \in \mathcal{L}(X, X)$ where $\mathcal{L}(X, X)$ is the space of bounded linear operators from X to itself. For each n let x_n be a point such that $P_n(Ax_n - b) = 0$. If $x_n \to x$, then $Ax = b$.

   b. Let X be a Banach space and $P_1, P_2, \cdots$ projections on X such that $P_n x \to x$ for every x. Suppose that A is an invertible element of $\mathcal{L}(X, X)$. For each n, let x_n be a point such that $P_n x_n = x_n$ and $P_n(Ax_n - b) = 0$. Prove or disprove that the sequence $[x_n]$ necessarily converges to the solution of the equation $Ax = b$.

8. a. Define what is meant by a distribution.

   b. Let g be continuously differentiable on $R \backslash \{0\}$. Assume that g has a jump discontinuity at the origin with jump $\sigma_0 = g(0+) - g(0-)$. Let g' denote the distributional derivative of g while Dg denotes the derivative of g. Show that $\forall f \in C_c^\infty(R)$, $g'(f) = \int (Dg)(t) f(t) dt + \sigma_0 f(0)$.

   c. Let ϕ be in $C_c^\infty(R)$. Prove that if there exists a multi-index α for which $D^\alpha \phi = 0$, then $\phi = 0$. Hint: Do the cases $|\alpha| = 0$ and $|\alpha| = 1$ first.

9. Explain how Fourier transform methods help in solving *either* of the following PDEs.

   a. The heat equation $u_{xx} = u_t$ with initial condition $u(x, 0) = f(x)$, $-\infty < x < \infty$.

   b. The Helmholtz equation $\Delta u - u = f$, where Δ is the Laplacian on R^n.