

Applied Mathematics Qualifying Exam

January 12, 2007

Instructions: *Do any four of problems 1 through 5 in part 1, and any three of problems 6 through 9 in part 2.. Show all of your work clearly. Please indicate of the problems you are skipping.*

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem in such a way that it becomes trivial.

Part 1. Do any four problems.

1. Let \mathcal{H} be a complex Hilbert space, with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ being the inner product and norm.
 - (a) Define these terms (all for \mathcal{H}): bounded linear operator, adjoint of a bounded linear operator, compact linear operator.
 - (b) Let $\lambda \in \mathbb{C}$ be fixed. If $K : \mathcal{H} \rightarrow \mathcal{H}$ is a compact linear operator, show that the range of the operator $L = I - \lambda K$ is closed.
 - (c) Determine the values of λ for which $u = f + \lambda K u$ has a solution for all $f \in \mathcal{H}$, given that $Ku(x) = \int_0^\pi \cos(x-t)u(t)dt$.
2. Let \mathcal{H} and K be as in problem 1. If $\{P_n\}$ is a sequence of orthogonal projections on \mathcal{H} such that $\lim_{n \rightarrow \infty} \|P_n u - u\| = 0$ for all $u \in \mathcal{H}$, then show that $\lim_{n \rightarrow \infty} \|P_n K - K\|_{op} = 0$. Explain how this is used in the Galerkin method for solving $u - K u = f$, assuming $I - K$ is invertible.
3. State the Weierstrass Approximation Theorem and sketch a proof of it.
4. State the Courant-Fischer minimax principle. Use it to show that λ_2 , the middle eigenvalue of A below, satisfies $\lambda_2 \leq -1/3$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{pmatrix},$$

5. Suppose that $Lu := -u'' + e^x u$, with domain $D = \{u \in L^2[0, 1] \mid u'' \in L^2[0, 1], u(0) = 0, u(1) = 0\}$.
- Show that L is self-adjoint on D . (That is, $L^* = L$ in the inner product of $L^2[0, 1]$.)
 - Let $H = \{u \in L^2([0, 1] \mid u' \in L^2[0, 1])\}$, together with an inner product $\langle u, v \rangle := \int_0^1 (u'v' + e^x uv) dx$ and norm $\|u\| = \sqrt{\langle u, u \rangle}$. Given that H is a Hilbert space, show that $Lu = f$, $f \in L^2[0, 1]$ has a unique weak solution.
 - Let $S^h(1, 0)$ be the space of linear splines on $[0, 1]$, with $h = 1/n$. Explain how to use the Rayleigh-Ritz method and the finite element space to approximately solve $Lu = f$.

Part 2. Do any three problems.

6. A mass m is constrained to move on the surface of a unit sphere, subject to a potential $U = U(\theta, \varphi)$. The angles θ and φ are the colatitude and longitude, respectively. The Lagrangian for this system is $L = T - U$, where $T = \frac{m}{2}(\dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2)$.
- State Hamilton's principle and use it to write out the equation of motion for the mass.
 - Find the momenta p_θ and p_φ conjugate to θ and φ , respectively, and also the Hamiltonian $H(\theta, \varphi, p_\theta, p_\varphi)$ for the system.
 - Write down Hamilton's equations for the system. Use them to show that H is a constant of the motion.
7. Consider the Γ -function, which is defined by $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$. Use the method of Laplace to establish Stirling's formula,

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} e^{-x} (1 + \mathcal{O}(x^{-1})), \quad x \rightarrow \infty.$$

8. Let \mathcal{S} be Schwartz space and \mathcal{S}' be the space of tempered distributions.
- Define \mathcal{S} and give the semi-norm topology for it. In addition, define \mathcal{S}' .
 - Explain how one defines the Fourier transform of a tempered distribution.
 - Show that if $T \in \mathcal{S}'$, then $\widehat{T^{(k)}} = (-i\omega)^k \widehat{T}$, where $k = 1, 2, \dots$ (Use the Fourier transform convention $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{i\omega t} dt$.)
 - Let $T(t) = (1 - |t|)_+$, where $(t)_+ = \frac{1}{2}(t + |t|)$. Show that $T''(t) = \delta(t + 1) - 2\delta(t) + \delta(t - 1)$. Use (8c) to find \widehat{T} .
9. Consider the operator $Lu = -u''$ defined on functions in $L^2[0, \infty)$ having u'' in $L^2[0, \infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0, \infty) \mid u'' \in L^2[0, \infty) \text{ and } u'(0) = 0\}.$$

- Find the Green's function $G(x, \xi; z)$ for $-G'' - zG = \delta(x - \xi)$, with $G_x(0, \xi; z) = 0$. (This is the kernel for the resolvent $(L - zI)^{-1}$.)
- Employ the spectral theorem to obtain the sine transform formulas,

$$F(\mu) = \frac{2}{\pi} \int_0^\infty f(x) \cos(\mu x) dx \text{ and } f(x) = \int_0^\infty F(\mu) \cos(\mu x) d\mu.$$