# Applied Mathematics Qualifying Exam

January 1997

**Instructions:** Attempt any 6 of the following 7 questions. Clearly show all of your work.

**1.** Let $T : L^2(0,1) \to L^1(0,1)$ be a linear operator. Assume that

$$T^{-1} : L^1(0,1) \to L^2(0,1)$$

exists and is bounded. Let $\alpha \in L^\infty(0,1)$ and $g \in L^1(0,1)$. Prove that there exists a constant $C > 0$ such that if $\|\alpha\|_\infty \le C$ then the equation

$$Tf + \alpha f^2 = g$$

$\|f - T^{-1}g\|_2$

has a unique solution $f$ in the set $B = \{f \in L^2(0,1) : \|f\|_2 \le 1\}$.

**2.** Let $H$ be a real Hilbert space and let $b$ be a bounded, coercive bilinear form on $H \times H$. Let $B : H \to H'$ be the operator defined by $(Bx)(y) = b(x,y)$ (where $H'$ denotes the space of bounded linear functionals on $H$). The *Lax-Milgram* Theorem asserts that $B^{-1} : H' \to H$ exists and is bounded.

Use the Riesz Representation Theorem for Hilbert spaces to prove the Lax-Milgram Theorem.

**3.** Let $H$ be a real separable Hilbert space, and let $\{\phi_n\}_{n=1}^\infty \subset H$ be an orthonormal basis for $H$. Let $A : H \to H$ be a bounded linear operator for which

$$A\phi_n = a_n \phi_n$$

for all $n$, where each $a_n$ is a real number. Assume that $a_n \to 0$ as $n \to \infty$.

**a.** Prove that $A$ is compact and self-adjoint.

**b.** What condition on $y \in H$, in terms of the null space of the operator $I - A$, guarantees that the problem

$$(I - A)x = y$$

has a solution $x \in H$? Why?

c. Assuming that $y$ satisfies the condition in part b, prove that the equation

$$(I - A)x = y$$

has a unique minimum-norm solution $x \in H$.

d. Suggest a constructive method for calculating finite-dimensional approximations $x_N$ of $x$ and show that $x_N \to x$ as $N \to \infty$ for your method.

4. Let $J : L^2(0,1) \to R$ be a continuous (but not necessarily linear) function. Given $f \in L^2(0,1)$, denote $Tf = u$, where $u \in H_0^1(0,1)$ satisfies

$$\int_0^1 u'v' + \int_0^1 uv = \int_0^1 fv, \quad \text{for all } v \in H_0^1(0,1).$$

Let $A = \overline{\{Tf : \|f\|_2 \leq 1\}}$, where the closure is taken with respect to $L^2(0,1)$. Prove that there exists a solution $u$ to the problem

$$\inf_{u \in A} J(u).$$

5. Let $< \cdot, \cdot >$ denote the standard inner product on the Sobolev space $H^1(0,\pi)$, and denote $S = \{\sin(nx)\}_{n=1}^{\infty}$. Show that $S$ is an orthogonal basis for $H_0^1(0,\pi)$ with respect to $< \cdot, \cdot >$. Extend $S$ to an orthogonal basis for $H^1(0,\pi)$.

6. Let $X$ be a Banach space and suppose $F : X \to X$ is continuously Frechet differentiable. Suppose $x \in X$ such that $F(x) = 0$ and $F'(x)$ is nonsingular. Prove there exists a number $\alpha > 0$ such that if $\|x_0 - x\| < \alpha$, then the sequence $\{x_n\}_{n=1}^{\infty}$ given by

$$F'(x_{n-1})y = -F(x_{n-1})$$

$$x_n = x_{n-1} + y$$

is well defined and $x_n \to x$ as $n \to \infty$.

7. Let $A$ be a real nonsingular $n \times n$ matrix and suppose the columns of $A$ are denoted by $\{c_1, c_2, \ldots, c_n\}$. Suppose $\{q_1, q_2, \ldots, q_n\}$ are the orthonormal vectors obtained by applying the Gram-Schmidt process to the vectors $\{c_1, c_2, \ldots, c_n\}$. Let $Q$ be the $n \times n$ matrix whose columns are given by $\{q_1, q_2, \ldots, q_n\}$. What can be said about the matrix $Q^T A$? Prove your assertion.