# Applied Mathematics Qualifying Exam

May 2000

**Instructions:** Attempt seven of the following questions. Clearly show all of your work.

1. Let $a = (a_1, a_2, \ldots, a_n, \ldots)$ be a bounded sequence and let $1 \le p \le \infty$.

   (a) For $x = (x_1, \ldots, x_n, \ldots) \in \ell^p$, i.e. $\sum_{n=1}^{\infty} |x_n|^p < \infty$, show that

   $$f(x) = (a_1 x_1, \ldots, a_n x_n, \ldots) \in \ell^p.$$

   (b) Prove that $f\colon\ \ell^p \to \ell^p$ as defined in part (a) is continuous.

   (c) Show that $f$ is compact if and only if $\lim_{n\to\infty} a_n = 0$.

2. Let $A$ be a self-adjoint operator on a Hilbert space $H$. Prove the following:

   (a) $\|A\| = \sup_{\|x\|\le 1} |(Ax, x)|$.

   (b) $\|A^n\| = \|A\|^n$ for all integers $n \ge 1$. *Hint:* For $\|x\| \le 1$ and $m \in \mathbb{N}$

   $$(*) \qquad \|A^m x\|^2 = (A^{m+1}x, A^{m-1}x) \le \|A^{m+1}x\|\ \|A^{m-1}x\|.$$

   Prove $(*)$ then prove (b) by induction on $n$.

3. Let $X := C[a, b]$, where $-\infty < a < b < \infty$ and $\|u\| := \max_{a\le x\le b} |u(x)|$.

   (a) Let $\alpha \in \mathbb{R}$ and $f \in X$ be given. Show that the nonlinear integral equation

   $$(*) \qquad u(x) = \alpha \int_a^b \cos u(x) dx = f(x), \qquad x \in [a, b]$$

   has a solution $u \in X$.

   (b) Let $\alpha \in \mathbb{R}$ with $|\alpha|(b - a) < 1$. Show that $(*)$ has a unique solution $u \in X$.

   (c) Propose a convergent algorithm to obtain the solution when

   $$|\alpha|(b - a) < 1.$$

4. For $0 < \alpha \leq 1$, let $C^{0,\alpha}[a,b]$ denote the set of Hölder continuous functions $u\colon\ [a,b] \to \mathbb{R}$; i.e.

$$|u(x) - u(y)| \leq C|x-y|^{\alpha}, \qquad \forall\ x, y \in [a,b].$$

Set $\|u\|_{\alpha} := \max_{a \leq x \leq b} |u(x)| + \sup_{\substack{a \leq x \leq b \\ a \leq y \leq b \\ y \neq x}} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}}$.

(a) Show that $(C^{0,\alpha}[a,b], \|\cdot\|_{\alpha})$ is a compact subset of $C[a,b]$.

(b) Let $H_0^1[a,b]$ be the Sobolev space $\{u \in L^2, u' \in L^2, u(a) = u(b) = 0\}$ with norm $\|u\|^2 = |u|_{L^2}^2 + |u'|_{L^2}^2$. Show that $\exists\ \alpha \in (0,1)$ that $H_0^1[a,b] \subset C^{0,\alpha}[a,b]$. Find the largest possible $\alpha$ such that this embedding holds.

5. Let $f \in L^2(0,1)$ and suppose that $f(x) = \sum_{n=0}^{\infty} a_n e^{i2\pi nx}$; $x \in [0,1]$ with $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.

(a) Find the Fourier series of the tempered distribution $x \mapsto f^{(k)}(x)$, $1 \leq k \leq N$.

(b) Assume that $x \mapsto f^{(k)}(x) \in L^2(0,1)$. What can you say about the sequence $(a_n)$? (You need to find an equivalent condition on $(a_n)$ so that $f^{(k)} \in L^2(0,1)$.)

(c) If $f \in L^2(0,1)$, $f' \in L^2(0,1)$ and $a_0 = 0$, prove

$$|f|_{L^2}^2 \leq 4\pi^2 |f'|_{L^2}^2.$$

(d) Assume that $a_0 = 1$. Does there exist a constant $C_0$ such that

$$|f|_{L^2}^2 \leq C_0 |f'|_{L^2}^2 \qquad \forall \{f \in L^2, f' \in L^2, a_0(f) = 1\}?$$

6. Consider the boundary value problem

$$\text{BV} \begin{cases} \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) - [\lambda + q(x)]u(x) = f(x), & a < x < b \\ u(a) = u(b) = 0. \end{cases}$$

where $q, f \in C([a,b])$ and $p \in C^1[a,b]$ are given functions with $p(x) > 0$ for $a \leq x \leq b$. Let $X = \{u \in C^2([a,b]);\ u(a) = u(b) = 0\}$ and let $T\colon\ X \to C([a,b])$ defined by

$$Tu = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) - q(x)u(x).$$

Assume that the equation $Tv = h$ (where $h$ is given) has at most one solution $v \in X$.

(a) Write (BV) in the form of an integral equation

(b) Discuss the existence of solutions to (BV).

7. Let $X = L^2(0,1)$ and let $T\colon X \to X$ be defined by

$$(Th)(x) = \int_0^x g(x,t)h(t)dt + \int_x^1 g(t,x)h(t)dt$$

where

$$g(t,x) = \cot(1)\cos x \cos t + \sin t \cos x.$$

(a) Prove that $T$ is a compact operator.

(b) Prove that $T$ is self-adjoint.

(c) Let $A$ be the Friedrichs extension on $X$ of the operator $B = -\frac{d^2}{dx^2}$ with

$$D(B) = \left\{u \in C^2([0,1]) : \quad u(0) = u(1) = 0\right\}.$$

Prove that $(I - A)T = T(I - A) = I$, where $I$ is the identity operator on $L^2(0,1)$.

8. Let $X = C[0,\pi]$, with $\|u\| = \max_{0 \le x \le \pi} |u(x)|$. Define

$$\begin{aligned} T_1\colon \quad & X \longrightarrow X \\ & u \longrightarrow T_1 u; \quad (T_1 u)(x) = \int_0^x \cos(x-t)u(t)dt. \end{aligned}$$

(a) Study the existence and uniqueness of solutions in $C[0,\pi]$ of the equation

$$(I - \lambda T_1)u = g, \quad \text{where } g \text{ is given in } C[0,\pi].$$

(b) Let $Y = L^2(0,1)$ and define $T_2\colon\ Y \to Y$ by

$$(T_2 u)(x) = \int_0^1 \cos(x-y)u^2(y)dy, \qquad 0 < x < 1.$$

(i) For $g \in L^2(0,1)$, study the existence and uniqueness of solutions to

$$T_2 u + u = g.$$

(ii) Same question but restrict $u$ to the set $\{u \in L^2; \|u - g\|_{L^2} \le 1\}$.