APPLIED MATHEMATICS QUALIFIERS–May 2002

You should attempt all 7 problems on the exam. Clearly show all your work.

Policy on misprints. The qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

1.

a. State the Babuska-Lax-Milgram Theorem.

b. Suppose a problem has been recast so that it is equivalent to solving: given a specific point $z \in V$, find $w \in U$ such that for all $v \in V$, $B(w, v) =< z, v >$ where B is a bilinear form satisfying the hypotheses of the Lax-Milgram theorem. Discuss how the Galerkin procedure can be used to solve this problem on a succession of finite-dimensional subspaces $U_n \subset U$ and $V_n \subset V$.

c. Prove the following: Let H be a Hilbert space, K a compact, linear operator on H and $\{P_n\}$ a sequence of projections with the property that $||P_n u - u|| \to 0$ as $n \to \infty$ for every $u \in H$. Show that if $(I - \lambda K)^{-1}$ exists, then u_n, the solution of $(I - \lambda P_n K)u_n = P_n f$, converges to the solution of $(I - \lambda K)u = f$ as $n \to \infty$.

2.

a. Define Frechet derivative and determine the Frechet derivative of the map $f(x) = \int_0^1 |x(t)|dt$ for those functions $x(t) \in C[0, 1]$ having at most a finite number of zeros in [0,1].

b. Let f be a function from R to R that satisfies the inequalities $f' > 0$ and $f'' > 0$. Prove that if f has a zero, then the zero is unique, and Newton's iteration, started at any point, converges to the zero.

c. Indicate how one would use a vector-valued Newton's method to solve the non-linear system of equations

$$x - y + 1 = 0$$
$$x^2 + y^2 - 4 = 0$$

Starting with the vector $\mathbf{u_0} = (0, 2)$, compute at least one iteration with this method.

3.

a. State a Lagrange multiplier theorem for minimizing (or maximizing) a "constrained" function.

b. Let A and B be Hermitian operators on a real Hilbert space. Prove that the stationary values of $< x, Ax >$ on the manifold $< x, Bx >= 1$ are necessarily numbers λ for which $A - \lambda B$ is *not* invertible.

c. State the Euler necessary condition for minimizing $\int_a^b F(x, y(x), y'(x))dx$ subject to the constraints $y(a) = \alpha, y(b) = \beta$ and use it to find the equation of an arc of minimal length joining two points in the plane.

4. Let $f \in L^2(0,1)$ and suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n e^{i2\pi nx}; x \in [0,1], \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

a. Find the Fourier series of the tempered distribution $x \to f^{(k)}(x), 1 \leq k \leq N$.

b. Assume that $x \to f^{(k)}(x) \in L^2(0,1)$. What further conditions are imposed on the sequence (a_n)?

c. If $f \in L^2(0,1)$, $f' \in L^2(0,1)$ and $a_0 = 0$, prove

$$|f|_{L^2}^2 \leq \frac{1}{4\pi^2} |f'|_{L^2}^2 \ .$$

d. Assume that $a_0 = 1$. Does there exist a constant C_0 such that

$$|f|_{L^2}^2 \leq C_0 |f'|_{L^2}^2 \quad f \in L^2, f' \in L^2, a_0(f) = 1 \ ?$$

5. Given the two-point boundary value problem

$$-u''(x) = f(x), \quad u(\alpha) = 0, \quad u(\beta) = 0$$

a. Find the Green's function G(x,y) for this equation.

Next, assume $\alpha = 0, \beta = 1$ and let U_y be the distribution that corresponds to the function $U_y(x) := G(x,y)$ for all $x \in [0,1]$. Let $\mathcal{D}(0,1)$ denote the space of test function which vanish at the endpoints $\{0,1\}$ and let δ_y denote the distribution "point evaluation at y".

b. Define what it means for two distributions to be equal on $\mathcal{D}(0,1)$.

c. Show that $-U_y'' = \delta_y$ on $\mathcal{D}(0,1)$.

6. Let $K : L^2[0,1] \to L^2[0,1]$be a compact, self-adjoint operator with $\{\phi_k\}_{k=1}^M$ the normalized eigenfunctions of K associated with the *nonzero* eigenvalues λ_k of K. Assume also that $\lambda_{k+1} \leq \lambda_k \quad \forall k \geq 1$. For a given $f \in L^2[0,1]$, discuss the existence (or nonexistence) of solutions to

$$Ku = \lambda u + f$$

where $b_m :=< f, \phi_m >$ represents the m^{th} Fourier coefficient of f with respect to $\{\phi_k\}_{k=1}^M$. In particular, what happens in the cases:

a. $\lambda \neq 0$, and λ does not coincide with any of the eigenvalues of K.

b. $\lambda \neq 0$ and $\lambda = \lambda_m$ for some fixed index m. Furthermore assume λ_m is a simple eigenvalue and that $b_m \neq 0$.

c. Same assumptions as in part b, except that $b_m = 0$.

7. Do the following three parts.

a. Prove the following theorem: Suppose that X and Y are Banach spaces over K and the linear operator A: $D \subseteq X \to Y$ satisfies $||Au|| \leq C||u|| \quad \forall u \in D$ where $C \geq 0$ is a constant where D is a dense linear subset of X. Show that the operator A can be uniquely extended to a linear continuous operator $A : X \to Y$ with $||A|| \leq C$ and if, in addition, A is compact on D, then so is the extended operator $A : X \to Y$.

b. Define under which conditions a function has a Fourier transform and define what it is.

c. Compute the Fourier transform (and verify!) of $f * g$ where

$$f * g(x) := \int_R f(x-t)g(t)dt$$

under the assumption that f, g and $f * g$ are all in $L^1(R)$.