

**APPLIED ANALYSIS QUALIFYING EXAMINATION
MAY 2003**

Hand in all the problems that you attempt. Your grade will be based on your best 7 answers.

Policy on misprints. *The qualifying examination committee tries to proof-read the examinations as carefully as possible. Nevertheless, there may be a few misprints. If you are convinced that a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.*

Q1. (a) State the inversion theorem for the Fourier transform on \mathbb{R} , stating carefully the hypotheses on f and \hat{f} .

(b) Let f be a function on \mathbb{R} such that f has compact support, is continuous on $(-\infty, a) \cup (a, \infty)$ for some $a \in \mathbb{R}$ and

$$\lim_{x \rightarrow a-} f(x) \text{ and } \lim_{x \rightarrow a+} f(x)$$

exist and are unequal. Prove that $\hat{f} \notin L^1(\mathbb{R})$. Verify by direct calculation that this is so for the function

$$f(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$

Q2. (a) Define the Fréchet derivative of a function $F : X \rightarrow Y$, where X and Y are Banach spaces.

(b) Let $T : C[0, 1] \rightarrow C[0, 1]$ be a bounded linear operator. Prove that the equation $T(f^2) + f = g$ has a solution for all $g \in C[0, 1]$ of sufficiently small norm. State any theorem that you use.

Q3. Consider the partial differential equation

$$2u_{xx} - 3u_{xy} + u_{yy} = 0.$$

Find an invertible 2×2 matrix A such that the change of variables

$$\begin{pmatrix} s \\ t \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

transforms the equation to $u_{st} = 0$, and hence find the general solution.

Q4. (a) State the implicit function theorem in a Banach space setting.

(b) Prove that the equations

$$\begin{aligned} x + y^2 + z^3 &= 3 \\ x^4 + y^5 + z^6 &= 3 \end{aligned}$$

can be solved for each pair of variables in terms of the remaining variable in a neighborhood of $(1, 1, 1)$.

Q5. (a) Let $F(s, t)$ be a function on \mathbb{R}^2 such that $F_{tt}(s, t) \geq 0$ for all s and t , and consider the problem of minimizing

$$\int_0^1 F(x, y'(x)) dx$$

subject to the conditions $y(0) = \alpha$ and $y(1) = \beta$. Prove that a solution of Euler's equation gives a global minimum.

(b) Find the function $y \in C^1[0, 1]$ which minimizes

$$\int_0^1 \sqrt{(1+x^2)^2 + y'(x)^2} dx$$

subject to $y(0) = 0$ and $y(1) = 4$, proving that it does give the minimum value.

Q6. Let $f \in C([0, 1]^2)$ and $\xi \in C[0, 1]$ be given. For each $\varepsilon > 0$, let $T_\varepsilon : C[0, 1] \rightarrow C[0, 1]$ be defined by

$$T_\varepsilon g(x) = g(x)/2 + \int_0^1 f(x, y)g(y)^2 dy - \varepsilon\xi(x), \quad g \in C[0, 1].$$

If ε is sufficiently small, prove that there exists $k > 0$ such that T_ε maps the closed ball $B_{k\varepsilon}$ of radius $k\varepsilon$ centered at 0 to itself and is a contraction on this set. With f as above and $\eta \in C[0, 1]$ given, prove that the equation

$$g(x) - 2 \int_0^1 f(x, y)g(y)^2 dy = \varepsilon\eta(x)$$

has a solution $g \in C[0, 1]$ for ε sufficiently small.

Q7. Let $\mathcal{D}(\mathbb{R})$ be the space of test functions on \mathbb{R} and let $\mathcal{D}'(\mathbb{R})$ be the space of distributions.

(a) Let $M = \{\phi'(x) : \phi \in \mathcal{D}(\mathbb{R})\}$. Prove that a test function ψ is in M if and only if $\int_{-\infty}^{\infty} \psi(x) dx = 0$.

(b) If $T \in \mathcal{D}'(\mathbb{R})$, prove that there exists $S \in \mathcal{D}'(\mathbb{R})$ such that $S' = T$, and that S is unique up to distributions given by constant functions.

Q8. Justify the following steps in finding the general solution to the distributional differential equation

$$T' + T = \delta,$$

giving any relevant definitions:

- (i) Multiply by e^x ;
- (ii) Write the equation as $(e^x T)' = \tilde{H}'$ where H is the Heaviside function;
- (iii) T is the regular distribution associated to the function $e^{-x}(H(x) + c)$ where c is an arbitrary constant.

Q9. (a) State the spectral theorem for a self-adjoint compact operator T on a complex Hilbert space, and prove that if λ is not in the spectrum $\sigma(T)$ of T then

$$\|(T - \lambda I)^{-1}\| = (\text{dist}(\lambda, \sigma(T)))^{-1}.$$

(b) Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of self-adjoint compact operators on a complex Hilbert space which converges in norm to a compact operator K . Prove that each non-zero eigenvalue λ of K is the limit of a sequence $\{\lambda_{n_r}\}_{r=1}^{\infty}$ where each λ_{n_r} is an eigenvalue of K_{n_r} .