APPLIED ANALYSIS QUALIFYING EXAMINATION
MAY 2004

Hand in all the problems that you attempt. Your grade will be based on your best 7 answers

Policy on misprints. *The qualifying examination committee tries to proof-read the examinations as carefully as possible. Nevertheless, there may be a few misprints. If you are convinced that a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem so that it becomes trivial.*

1.

a. Let X be a Banach space and let $A : X \to X$ be a linear operator of norm less than 1. If I is the identity on X, prove that $I - A$ is invertible.

b. Suppose that the operator A satisfies a polynomial equation

$$\sum_{j=0}^{n} c_j A^j = 0$$

in which $c_0 \neq 0$. Prove that A is invertible and give a formula for its inverse.

c. Prove or disprove: If A is a bounded linear operator on a Banach space and if $||A^m|| < 1$ for some $m \geq 1$, then $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$.

2.

a. State the Contraction Mapping Principle.

b. Establish the following: Let v be continuous on $[a, b]$ and let K be continuous on the square $[a, b] \times [a, b]$. Then the integral equation

$$x(t) = \int_a^t K(s, t)x(s)\, ds + v(t)$$

has a unique solution in $C[a, b]$.

3.

a. Define the Schwartz space $\mathcal{S}$ on $\mathbf{R}$ and state the inversion theorem for $\mathcal{S}$.

b. Explain why, for $\phi \in \mathcal{S}$,

$$\int\int e^{-2\pi i x(y+z)}\phi(y)\,dydx = \phi(-z).$$

c. Let $f(x) = e^{-x}$ for $x \geq 0$ and let $f(x) = 0$ for $x < 0$. Find $\hat{f}$ and verify by direct integration that $f(x) = \int \hat{f}(y)e^{2\pi i x y}dy$.

4.

a. Let $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$ be two normed linear spaces. Define what it means to say "X embeds in Y".

b. Find the norm of the identity operator for the embedding of $C[a, b]$ into $L^2[a, b]$.

c. Define the Sobolev spaces $W^{k,p}(\mathbf{R})$ and show that the function

$$f(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 1 \end{cases}$$

belongs to $W^{0,p}(\mathbf{R})$ but not $W^{1,p}(\mathbf{R})$.

5.

a. Prove **one** of the following statements:
A. Every distribution on $\mathbf{R}$ is the derivative of another distribution.
B. Let T be any distribution on $\mathbf{R}$ such that $\partial T = 0$. Then T is $\tilde{c}$ for some constant c.

b. Let $\{f_j\}_{j=1}^{\infty}$ be a sequence of nonnegative functions in $L^1(\mathbf{R}^n)$ such that $\int f_j = 1$ for each $j \geq 1$ and such that

$$\lim_{j\to\infty} \int_{|x|\geq r} f_j = 0$$

for all $r > 0$. Then $\tilde{f}_j \to \delta$ (the Dirac distribution) as $j \to \infty$.

6.

a. Let X be a normed linear space and let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence which converges to x. Let $\sigma_n = (x_1 + \cdots + x_n)/n$ for $n \geq 1$. Show that $\{\sigma_n\}_{n=1}^{\infty}$ also converges to x.

b. Let M and N be two closed subspaces of a Banach space X and further assume that N is finite dimensional. Show that the space $M+N$ is closed. (Hint: First do the special case that $M \cap N = 0$).

7.

a. State **one** of these theorems: the Implicit Function Theorem, the Inverse Function Theorem or the Surjective Mapping Theorem.

b. Let f be a continuous map from an open set Ω in a Banach space X into a Banach space Y, and suppose that $f'(x)$ exists and is continuous in Ω. If $x_0 \in \Omega$ is a point where $f'(x_0)$ is invertible, then prove that f is one-to-one in some neighborhood of x_0.

8.

a. Let X and Y be Banach spaces and $f : X \to Y$ a map from X to Y. Define the Fréchet derivative $f'(x_0)$ of f at $x_0 \in X$.

b. Let $p \in (2, \infty)$ be fixed. Let $X = L^p(\mathbf{R})$, let $f : X \to \mathbf{R}$ be defined by $f(x) = \|x\|_p$ for $x \in X$, and let $x_0 \in X$ be nonzero. Show that $f'(x_0)$ is the linear functional defined by the function

$$x_0(\cdot)|x_0(\cdot)|^{p-2}/||x_0||_p^{p-1} \in L^q(\mathbf{R}),$$

where $1/p + 1/q = 1$.

(Hint: it is easier to work with $f^p(x)$ and use the chain rule).

9. Find the Green's function for the problem

$$x'' - x' - 2x = y, \quad x(0) = 0 = x(1).$$