# Applied Mathematics Qualifying Exam

May 2005

**Do any 7 of the 9 problems in this exam. Clearly show all of your work.**
**Please indicate which 2 of the 9 problems you will skip in your work.**

**Policy on Misprints.** The Qualifying Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do not interpret the problem in such a way that it becomes trivial.

**1.** (a). Let $P_1, P_2, \cdots ,$ be a sequence of projections on a normed linear space $X$. Suppose that $P_{n+1}P_n = P_n$ for all $n$ and that the union of the ranges of these projections is dense in $X$. Suppose further that $sup_n ||P_n|| < \infty$. Prove that $P_n x \to x$ for all $x \in X$.

(b). Let $\{\phi_1, \phi_2, \cdots , \phi_n\}$ be a linearly independent set in $X^*$. Is there a projection $P : X \to X$ having rank $n$ of the form

$$Px = \sum_1^n \phi_i(x) v_i?$$

(Here $\{v_i\}_1^n$ is some collection to be specified of elements of $X$).

**2.** (a). State and prove the Contraction Mapping Theorem.

(b). Prove the following: Let $S$ be an interval of the form $S = [0, b]$ and $f$ be a continuous map of $S \times \mathbb{R}$ to $\mathbb{R}$. Assume a Lipschitz condition in the second argument of $f$:

$$|f(s, t_1) - f(s, t_2)| \leq \lambda |t_1 - t_2|$$

where $\lambda$ is a constant depending only on $f$. Then the initial-value problem

$$x' = f(s, x(s)) \quad x(0) = \beta$$

has a unique solution in $C(S)$. Hint: Introduce a new norm in $C(S)$ by defining

$$||x||_w = \sup_{s \in S} |x(s)| e^{-2s\lambda}$$

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**3.** (a). Let $X$ and $Y$ be Banach spaces. Let $x_0 \in X$ and $f : X \to Y$. Define the Frechet derivative of f at $x_0$ (if it exists).

(b). Let $f : \mathbb{R}^n \to \mathbb{R}$. Explain the connection between the Frechet derivative $f'(x_0)$ and the gradient vector $\nabla f(x_0)$.

(c). Let $f : C[0,1] \to \mathbb{R}$ be defined by $f(x) = \int_0^1 |x(t)|dt \quad \forall x \in C[0,1]$. Find the Frechet derivative of $f$ at $x \in C[0,1]$ where it exists.

**4.** (a). Let $X$ be a Hilbert space having a countable orthonormal basis $\{u_1, u_2, \cdots, \}$. Define an operator A by the equation $Ax = \sum_{n=1}^{\infty} \langle x, u_n \rangle u_{n+1}$. What are the eigenvalues of $A$? Is $A$ compact? Is $A$ Hermitian? What is the norm of $A$?

(b). Same questions as (a) for $Ax = \sum_{n=1}^{\infty} \alpha_n \langle x, u_n \rangle u_n$ in which $\{\alpha_n\}$ is some prescribed bounded real sequence. Find the conditions under which $A^{-1}$ exists as a bounded linear operator.

(c). Same questions as (a) for the operator $Ax = \sum_{n=1}^{\infty} \langle x, u_{n+1} \rangle u_n$.

**5.** (a). Let $X$ and $Y$ be Banach spaces and $f : X \to Y$. In this setting identify the problem that Newton's method addresses and the corresponding algorithm used to (attempt to) solve this problem.

(b). Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x - y + 1, x^2 + y^2 - 4)$. Apply Newton's algorithm to compute $u_1$ and $u_2$ given the starting point $u_0 = (0, 2)$.

**6.** Let $\{u_1, ..., u_n\}$ and $\{v_1, ..., v_n\}$ be two sets in $L^2[0,1]$. Let $k(s,t) = \sum_{i=1}^{n} u_i(s) v_i(t)$ and

$$(Kx)(t) = \int_0^1 k(s,t) x(s) ds \quad \forall x \in L^2[0,1].$$

For $b \in L^2[0,1]$, find a solution expression to the problem $\min_{x \in L^2[0,1]} \|Kx - \lambda x - b\|$ where

(a). $\lambda = 0$;

(b). $\lambda \neq 0$ is not an eigen-value of $K$.

**7.** Let $\Omega \subset \mathbb{R}^n$ be an open set and $W^{k,2}(\Omega)$ be the Sobolev space defined by

$$W^{k,2}(\Omega) = \left\{ f \in L^2(\Omega): \text{ there exists } g_\alpha \in L^2(\Omega) \text{ s.t. } \partial^\alpha \tilde{f} = \tilde{g}_\alpha, \forall |\alpha| \leq k \right\}$$

where $\tilde{g}_\alpha$ denotes the distribution corresponding to $g_\alpha$.

(a). Prove that the formula

$$\langle f, g \rangle = \sum_{|\alpha| \leq k} \int_\Omega \partial^\alpha \tilde{f}(x) \partial^\alpha \tilde{g}(x) dx$$

defines an inner product in $W^{k,2}(\Omega)$;

(b). Verify that $W^{k,2}(\Omega)$ is a Hilbert space under the norm $\|f\|_{k,2} = \sqrt{\langle f, f \rangle}$.

**8.** Let $f: \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ function satisfying $f'(0) = 0, u^T f''(0) u > 0, \forall u \in \mathbb{R}^n, u \neq 0$. Assume that there is $\alpha > 0$, for each $u \in \mathbb{R}^n, \|u\| = 1$, there are unique $t_u > \alpha$ and $0 < t_u^0 < t_u$ such that

$$f'(t_u u) u = 0 \quad \text{and} \quad u^T f''(t_u^0 u) u = 0.$$

Define

$$\mathcal{N} = \{t_u u : u \in \mathbb{R}^n, \|u\| = 1, t_u > \alpha, f'(t_u u) u = 0\}$$
$$\mathcal{S} = \{t_u^0 u : u \in \mathbb{R}^n, \|u\| = 1, 0 < t_u^0 < t_u, u^T f''(t_u^0 u) u = 0\}.$$

Furthermore we assume that $u^T f''(t_u u) u < 0$ for each $t_u u \in \mathcal{N}$.

(a). Apply the Implicit Function Theorem to prove that $\mathcal{N}$ is a differentiable manifold;

(b). Prove that $\mathcal{S}$ is a continuous manifold and

$$\min_{u \in \mathcal{S}} \|f'(u)\| = r_{\mathcal{S}} > 0;$$

(c). Let $B$ denote the open set with boundary $\mathcal{S}$. If $u_0 \in B$ and $\|f'(u_0)\| > r_{\mathcal{S}}$, prove that any continuous path $\{u(\lambda) : 0 \leq \lambda \leq 1, u(0) = u_0\}$ defined by the homotopy

$$0 = \mathcal{H}(\lambda, u(\lambda)) \equiv \lambda f'(u(\lambda)) + (1 - \lambda)(f'(u) - f'(u_0))$$

stays in $B$.

**9.** (a). Define the Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$.

(b). Describe how Fourier transform can be used to solve the Helmholtz equation $\Delta u - u = f$ where $\Delta$ is the Laplacian in $\mathbb{R}^2$ and $f \in L^1(\mathbb{R}^2)$ is given;

(c). Assume that a solution is found in (b). Find a solution to the problem $\Delta u - c^2 u = f$ where $c > 0$.