Applied Mathematics Qualifying Exam

May 1996

Instructions: Attempt any 5 of the following 6 questions. Document all of your work.

1. Let $p \in C([0,1], R)$ and $f, q \in L^1((0,1), R)$ such that $p(x) > 0$ for all $0 \le x \le 1$ and $q \ge 0$ almost everywhere.

   a. Give a weak formulation for the boundary value problem

   $$\left\{ \begin{array}{c} -\frac{d}{dx}(p(x)u'(x)) + q(x)u(x) = f(x) \quad a.e.\ 0 < x < 1 \\ u(0) = 0, \quad u'(1) + u(1) = 0 \end{array} \right\} \tag{0.1}$$

   b. Prove that (0.1) has a unique weak solution.

   c. Consider the functions p and q fixed, and define a function

   $$T : L^2((0,1), R) \to L^2((0,1), R)$$

   via

   $$T(f) = u$$

   where u is the unique weak solution of (0.1). Prove that T is a compact, self-adjoint operator. What can you conclude?

2. Let $R_{n\times n}$ denote the space of real $n \times n$ matrices.

   a. Give a definition of the exponential of a matrix $A \in R_{n\times n}$ (denote $\exp(A)$ as the exponential of A).

   b. Let $A, B \in R_{n\times n}$ such that $AB = BA$. Prove $\exp(A+B) = \exp(A)\exp(B)$.

   c. Suppose $A \in R_{n\times n}$. Prove $\exp(A)$ is a nonsingular matrix and

   $$(\exp(A))^{-1} = \exp(-A). \tag{0.2}$$

   d. Let $A = \begin{pmatrix} 11 & -10 & -4 \\ 20 & -19 & -8 \\ -20 & 20 & 9 \end{pmatrix}$. Compute $\exp(A)$ and verify the formula (0.2) in part c.

3. Suppose $d, T > 0$ and $a, b \in R$. Let

$$u_0 \in L^2((0,1), R)$$

and

$$f \in L^2((0,1) \times (0,T), R).$$

a. Give a formula for the unique solution to the partial differential equation

$$\left\{ \begin{array}{cc} u_t = du_{xx} - u + f, & 0 < x < 1,\ 0 < t < T \\ u(0,t) = a, \quad u_x(1,t) = b & t > 0 \\ u(x,0) = u_0(x) & 0 < x < 1 \end{array} \right\} \tag{0.3}$$

b. Suppose $f(x,t) \equiv f(x)$. Discuss the limiting behavior of $u(x,t)$ as $t \to \infty$.

4. Let $\leq$ denote the natural partial ordering on R^n, and suppose $u, v \in R^n$ such that $u \leq v$. Denote

$$[u, v] = \{x \in R^n \mid u \leq x \leq v\}.$$

Suppose $F : R^n \to R^n$ is continuously differentiable and

$$\frac{\partial F_i(x)}{\partial x_j} \geq 0 \text{ for all } i \neq j \text{ and } x \in [u, v].$$

Furthermore, suppose

$$F(u) \geq u \text{ and } F(v) \leq v.$$

Prove there exists a vector $y \in [u, v]$ such that y is a fixed point of F. In addition, show there exists a value $M > 0$ sufficiently large so that if we define $y_0 = u$ and

$$y_k = \frac{1}{M+1}\left(F(y_{k-1}) + My_{k-1}\right)$$

for $k \in N$, then

$$u = y_0 \leq y_1 \leq \dots \leq y_k \leq v,$$

and the sequence $\{y_k\}_{k=1}^{\infty}$ converges to a fixed point of F.

5. Let $K \in L^2((0,1) \times (0,1), R)$ and define the operator

$$T : L^2((0,1), R) \to L^2((0,1), R)$$

via

$$T(u)(x) = \int_0^1 K(x,y)u(y)dy$$

a. Let $\{K_N\}_{N=1}^{\infty} \subseteq L^2((0,1) \times (0,1), R)$ such that $K_N \to K$ as $N \to \infty$, and define

$$T_N : L^2((0,1), R) \to L^2((0,1), R)$$

via

$$T_N(u)(x) = \int_0^1 K_N(x,y)u(y)dy.$$

Prove that $T_N \to T$ as $N \to \infty$.

b. Suppose $1 \notin \sigma(T)$. Prove there exists $M > 0$ such that if $N \geq M$ then $1 \notin \sigma(T_N)$.

c. Let M be given from part b and let $f \in L^2((0,1), R)$. Suppose

$$\{f_N\}_{N=M}^{\infty} \subseteq L^2((0,1), R)$$

such that $f_N \to f$ as $N \to \infty$. Let u_N denote the unique solution of

$$u_N - T_N(u_N) = f_N$$

for each $N \geq M$. Prove $u_N \to u$ as $N \to \infty$, where u is the unique solution of

$$u - T(u) = f.$$

6. Suppose $f : R^n \to R^n$ is a continuously differentiable function and $u_0 \in R^n$. Prove there exists a value $T > 0$ such that the initial value problem

$$\left\{ \begin{array}{ll} u'(t) = f(u(t)) & -T < t < T \\ u(0) = u_0 & \end{array} \right\}$$

has a unique solution $u \in C^1((-T,T), R^n)$. In addition, show that if $u_0 \geq 0$ and

$$f_i(v) \geq 0 \text{ whenever } v \geq 0 \text{ and } v_i = 0$$

then $u(t) \geq 0$ for all $0 < t < T$. (Note: v_i denotes the i-th component of v and $f_i(v)$ denotes the i-th component of $f(v)$.)