Applied Mathematics Qualifying Exam

May 1997

Instructions: Attempt any 6 of the following 7 questions. Clearly show all of your work.

1. Let X and Y be Banach spaces, let $B \subset X$ be an open convex set, and let $F : B \to Y$ be Fréchet differentiable over all of B. Prove the Mean Value Theorem: for any $x_1, x_2 \in B$, there exists a point $\tilde{x} \in B$ such that

$$\|F(x_2) - F(x_1)\| \leq \|DF(\tilde{x})(x_2 - x_1)\|,$$

where $DF(x)(y)$ denotes the Fréchet derivative of F at x in the direction y.

2. Let $B : H \to H$ be a compact, self-adjoint linear operator on the separable Hilbert space H. Assume that the null space of B is $\{0\}$. Use the Spectral Theorem to construct a sequence of bounded linear operators $A_n : H \to H$ such that for every fixed $x \in H$,

$$\|(I - A_n B)x\| \to 0, \quad \text{as } n \to \infty.$$

3. Let $\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ denote the Laplace operator, defined on smooth functions over $\mathbb{R}^n$. Given $s \geq 0$, let $H^s(\mathbb{R}^n)$ denote the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 \, d\xi,$$

where $\hat{u}$ denotes the Fourier transform of u. If $s < 0$, let $H^s(\mathbb{R}^n)$ denote the dual space of $H^{-s}(\mathbb{R}^n)$. Outline a procedure for extending $\triangle$ to the Sobolev spaces $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, and prove that

$$\triangle : H^s(\mathbb{R}^n) \to \mathrm{H}^{s-2}(\mathbb{R}^n)$$

for every $s \in \mathbb{R}$.

4. Let $a \in L^\infty(0,1)$ satisfy $a(x) \geq a_0 > 0$ a.e., where a_0 is constant. Let $q \in L^\infty(0,1)$. Prove that the problem

$$\inf_{u \in W_0^{1,3}(0,1)} J(u) = \int_0^1 (a|u'|^3 - qu)\, dx$$

has a solution. You may take as given the fact that $W_0^{1,3}(0,1)$ is reflexive.

5. Let $f \in L^2(0,1)$ and $a \in L^\infty(0,1)$ be given. Let $\{a_n\} \subset L^\infty(0,1)$. Assume that $a(x) \geq 0$ a.e., and $a_n(x) \geq 0$ a.e. for all n. For each n, let $u_n \in H_0^1(0,1)$ denote the unique weak solution to the problem

$$\begin{aligned} u_n'' - a_n u_n &= -f, \quad \text{on } (0,1), \\ u_n(0) = u_n(1) &= 0, \end{aligned}$$

and let $u \in H_0^1(0,1)$ denote the unique weak solution to

$$\begin{aligned} u'' - au &= -f, \quad \text{on } (0,1), \\ u(0) = u(1) &= 0. \end{aligned}$$

If $a_n \overset{*}{\rightharpoonup} a$, weak $*$ $L^\infty(0,1)$, show that there exists a subsequence of $\{u_n\}$ which converges weakly in $H_0^1(0,1)$ to u.

6. Discuss Newton's method. Use Newton's method to approximate a solution to the boundary value problem

$$\begin{aligned} -u''(x) &= \sin(u(x)) + 0.1, \quad 0 < x < 1, \\ u(0) &= u(1) = 0, \end{aligned}$$

by performing one iteration starting from $u_0 = 0$.

7. Prove that $u \equiv 0$ is the only continuous solution of the integral equation

$$u(x) + \int_0^1 \frac{u(y)}{u(y)^2 + 1 + x}\, dy = 0, \quad 0 \leq x \leq 1.$$