Applied Mathematics Qualifying Exam

May 1998

Instructions: Attempt any 6 of the following 7 questions. Show clearly all of your work.

1. Let $M_{m,n}$ denote the space of $m \times n$ complex matrices and $M_m = M_{m,m}$. Assume that the rank k matrix $A \in M_{m,n}$ has Singular Value Decomposition

$$A = V\Sigma W^*$$

where $V \in M_m$ and $W \in M_n$ are unitary and $\Sigma = [\sigma_{ij}] \in M_{m,n}$ has $\sigma_{ij} = 0$ for $i \neq j$, and $\sigma_{11} \geq \sigma_{22} \geq \ldots \geq \sigma_{kk} > \sigma_{k+1,k+1} = \ldots = \sigma_{qq} = 0$ where $q = \min[m, n]$. The *Moore-Penrose Generalized Inverse*, $A^\dagger$, is defined by

$$A^\dagger = W\Sigma^\dagger V^*$$

where $\Sigma^\dagger$ is the transpose of Σ in which the positive singular values of A are replaced by their reciprocals, i.e. $\Sigma^\dagger = [\sigma^\dagger_{ij}]$ with

$$\sigma^\dagger_{ij} = \begin{cases} \frac{1}{\sigma_{ij}} & \text{if } \sigma_{ij} \neq 0 \\ 0 & \text{if } \sigma_{ij} = 0. \end{cases}$$

a. Show that

  i. $AA^\dagger$ and $A^\dagger A$ are Hermitian;

  ii. $AA^\dagger A = A$;

  iii. $A^\dagger AA^\dagger = A^\dagger$.

b. A least-squares solution to the linear equations $A\mathbf{x} = \mathbf{b}$ is a vector $\mathbf{x}$ such that $\|\mathbf{x}\|_2$ is minimized among all vectors $\mathbf{x}$ for which $\|A\mathbf{x} - \mathbf{b}\|_2$ is minimal. Show that $\mathbf{x} = A^\dagger\mathbf{b}$ is a least-squares solution to $A\mathbf{x} = \mathbf{b}$.

2. Let $\eta \in \mathcal{R}^n$ be fixed and consider the function $g_\eta(x) = e^{2\pi i\eta\cdot x}$ defined over all of $\mathcal{R}^n$.

a. Explain in what sense the Fourier transform, $\hat{g}_\eta(\xi)$ for $\xi \in \mathcal{R}^n$, is a well-defined object, and then calculate it.

b. Explain in what sense the derivative, $\frac{\partial \hat{g}_\eta(\xi)}{\partial \xi_j}$, is a well-defined object for each $j = 1, 2, \ldots, n$ and calculate it.

3. Let $\mathcal{H}$ be a Hilbert space, and A a (possibly unbounded) linear operator acting in $\mathcal{H}$. Assume that the domain of A and the domain of A^* are both dense in $\mathcal{H}$. Determine which of the following statements are necessarily true; prove the true ones, and explain how the others (if any) could be violated. [A^* = adjoint of A, $S^\perp$ = orthogonal complement of S. ran = range (image), ker = kernel (nullspace).]

a. $\operatorname{ran} A \subseteq (\ker A^*)^\perp$

b. $(\operatorname{ran} A)^\perp \subseteq \ker A^*$

c. $(\ker A^*)^\perp \subseteq \operatorname{ran} A$

d. $\ker A^* \subseteq (\operatorname{ran} A)^\perp$

4. Let $\Omega \subset \mathbf{R}^2$ be the region $0 < x < \infty$, $0 < y < \pi$. Consider the boundary-value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } \Omega, \qquad u(x,0) = 0, \qquad u(x,\pi) = f(x), \qquad u(0,y) = g(y).$$

(For simplicity, assume that $f(0) = 0 = g(0) = g(\pi)$.)

a. Solve the problem (formally) by separation of variables (or equivalent Fourier analysis).

b. State some (fairly weak) technical conditions on f and g that guarantee that your formal solution is justified. Be as specific as you can about convergence properties of the Fourier expansions both on the boundary and in the interior. Feel free to provide a variety of technical hypotheses and a corresponding variety of conclusions.

(c) What complications might arise if we didn't assume that the data functions vanish at the corners? In particular, what if $f(0) = g(\pi) \neq 0$?

5. Given $f \in L^2(0,1)$, and $q \in L^\infty(-1,1)$, define the operator A by

$$(Af)(x) = \int_0^1 q(x-y) f^2(y)\,dy, \quad 0 < x < 1, \ a.e..$$

Given $g \in L^2(0,1)$, prove that there exists a positive constant C such that $\|q\|_\infty \le C$ implies that the equation

$$Af + f = g$$

has a unique solution in the set $\{f \in L^2(0,1) : \|f - g\|_2 \le 1\}$.

6. Let $C[0,\pi]$ denote the Banach space of continuous real valued functions defined on the interval $[0,\pi]$.

a. Let the operator $K : C[0,\pi] \to C[0,\pi]$ be defined by

$$K[f](x) = \int_0^x \cos(x-r) f(r)\,dr.$$

Discuss the questions of existence and uniqueness of solutions in $C[0,\pi]$ for the integral equation

$$(I - \lambda K)f = g$$

with $g \in C[0,\pi]$.

b. Repeat problem **2a** for the operator

$$K[f](x) = \int_0^\pi \cos(x-r) f(r)\,dr.$$

7. Consider the initial value problem

$$\begin{aligned} \mathbf{u}' &= \lambda \mathbf{F}(t, \mathbf{u}(t)) \\ \mathbf{u}(0) &= \mathbf{u}_0 \end{aligned}$$

where $\mathbf{u}(\cdot) : \ \mathcal{R}^+ \to \mathcal{R}^n$, $\mathbf{u}_0 \in \mathcal{R}^n$ and $\mathbf{F}(\cdot,\cdot) : \ \mathcal{R}^+ \times \mathcal{R}^n \to \mathcal{R}^n$ has the form

$$\mathbf{F}(\mathbf{u}) = \phi(|\mathbf{u}|) B \mathbf{u}$$

with B a constant $n \times n$ matrix and $\phi(r) = 2/\pi \arctan(\alpha r)$ for $\alpha > 0$.

a. Use the *Banach Fixed Point Theorem* to formulate a local existence theorem for solutions to this initial value problem in the Banach space $C^1([0,T];\mathcal{R}^n)$ of continuously differentiable functions from the interval $[0,T]$ into $\mathcal{R}^n$.

b. Redo part **a.** using the *Schauder Fixed Point Theorem*. Discuss whether or not this theorem is "better" than that in part **a.**.

c. What can one say about global existence in $C^1([0,\infty);\mathcal{R}^n)$?