Applied Mathematics Qualifying Exam

May 1999

Instructions: Attempt all six of the following questions. Clearly show all of your work.

1. For $k, n \in \{1, 2, 3, \ldots\}$, define

$$g(0, k, n) = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{otherwise} \end{cases}$$

and, for $t \neq 0$,

$$g(t, k, n) = e^{-k^2(k-n)^2/t^2}.$$

Prove that there exists $\epsilon > 0$ such that for each fixed t with $|t| < \epsilon$, and for every $v = (v_1, v_2, \ldots) \in l^\infty$, the problem

$$\sum_{k=1}^{\infty} g(t, k, n) u_k = v_n, \quad n = 1, 2, 3, \ldots$$

admits a unique solution $u = (u_1, u_2, \ldots) \in l^\infty$.

2. Let $Q \in \mathbb{R}^{n \times n}$ be a given orthogonal matrix (so $Q^T Q = QQ^T = I$). Let $\mathcal{S}$ denote the Schwartz class on $\mathbb{R}^n$. Define the *rotation operator* $R : \mathcal{S} \to \mathcal{S}$ by

$$(Rf)(x) = f(Qx), \quad \text{for } f \in \mathcal{S}.$$

Explain how R can be extended to the space $\mathcal{S}'$ of tempered distributions on $\mathbf{R}^n$, and prove that the Fourier transform $\mathcal{F}$ commutes with R on $\mathcal{S}'$, that is, $R\mathcal{F} = \mathcal{F}R$.

3. Let $\{f_j\}_{j=0}^{N} \subset L^\infty(0, 1)$ be a finite sequence of functions satisfying $\|f_j\|_\infty \leq 1/(N+1)$ for $j = 0, 1, \ldots, N$. Prove that the equation

$$u(x) = \sum_{j=0}^{N} \left(\int_0^1 u^j(y)\, dy \right) f_j(x), \quad \text{a.e. } x \in (0, 1),$$

has at least one solution $u \in B = \{u \in L^\infty(0, 1) : \|u\|_\infty \leq 1\}$.

4. Consider the problem

$$\begin{aligned} u^{(4)}(x) + u(x) &= f(x), \quad \text{for } x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) &= 0, \end{aligned}$$

where $u^{(4)}$ denotes the fourth derivative of u. Explain what is meant by a **generalized solution** to this problem, and prove that the problem admits a unique generalized solution u for each given $f \in L^2(0,1)$.

5. Let $g \in L^\infty(0,1)$, with $\|g\|_\infty \le 1/8$. Define $F(u) = v$, where v solves

$$\begin{aligned} -v'' &= (u-g)^2 \quad \text{on } (0,1), \\ v(0) = v(1) &= 0. \end{aligned}$$

Let $B_\alpha = \{u \in L^\infty(0,1) : \|u-g\|_\infty \le \alpha\}$. Prove that there exists $\alpha > 0$ such that the problem

$$F(u) = u$$

has a unique solution $u \in B_\alpha$.

6. (a) Let A denote an $m \times n$ real matrix with A^* its adjoint. Prove that

$$\text{Nul}[A^*] = \text{Ran}[A]^\perp$$

where $\text{Nul}[A^*]$ denotes the nullspace of A^*, and $\text{Ran}[A]^\perp$ denotes the orthogonal complement of the range of A.

(b) For $a, b \in \mathbb{R}^n$, $a \otimes b$ denotes the elementary tensor product of a and b, whose action as a linear transformation on $\mathbb{R}^n$ is defined by

$$(a \otimes b)(x) = a\langle b, x\rangle$$

for all $x \in \mathbb{R}^n$, where $\langle b, x\rangle$ denotes the usual inner product on $\mathbb{R}^n$. Using part (a), discuss the questions of existence and uniqueness of solutions to the system of equations

$$Ax = b$$

where $A \doteq c \otimes d + e \otimes f$ with $b, c, d, e, f \in \mathbb{R}^n$.