Combinatorics Qualifying Examination

Instructions

1. Please write your name on every page of your solutions.
2. There are 4 Combinatorics questions, worth a total of 50 points (2 questions worth 10 points each and 2 questions worth 15 points each). Your total grade for the Combinatorics section of this exam will be out of a maximum possible of 50 points.
3. Correctness of your solution and clarity of your writing are the determining factors in your grade. Elegance and originality are also welcomed by the graders.
4. You may use a calculator **only** to check numerical calculations.
5. If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

1. (10 points; 5,5)

   (a) Let

   $$f(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

   Prove that $f(x) = \frac{1}{\sqrt{1-4x}}$.

   (b) Prove the identity

   $$2^{2n} = \sum_{a+b=n} \binom{2a}{a}\binom{2b}{b}.$$

2. (10 points; 5,5) Let $\pi = (a_1, a_2, \ldots, a_n)$ be a permutation of the set $\{1, 2, \ldots, n\}$. Let $c_i = c_i(\pi)$ be the number of cycles of π of length i. Define the *type* of π to be the sequence $(c_1, \ldots, c_n)$.

   (a) Prove that the number of permutations of length n with type $(c_1, c_2, \ldots, c_n)$ is equal to $n!/1^{c_1}c_1!2^{c_2}c_2! \cdots n^{c_n}c_n!$.

   (b) Let $c(n,k)$ be the number of permutations on length n with exactly k cycles. Prove the numbers $c(n,k)$ satisfy the recurrence

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1),$$

   with the initial conditions $c(n,k) = 0$ if $n \leq 0$ or $k \leq 0$, except $c(0,0) = 1$.

3. (15 points; 5,5,5) Let $\pi = (a_1, a_2, \ldots, a_n)$ be a permutation of the set $\{1, 2, \ldots, n\}$. If $a_{r+1} = a_r + 1$, $1 \le r \le n-1$, then the pair (j_r, j_{r+1}) is called a succession of the permutation π.

Let S_n be the number of permutations of the set $\{1, 2, \ldots, n\}$ without a succession. Prove the following.

(a)

$$S_n = (n-1)! \sum_{k=0}^{n-1} (-1)^k \frac{n-k}{k!}.$$

(b) The number $S_{n,k}$ of permutations of the set $\{1, 2, \ldots, n\}$ with k successions is equal to

$$S_{n,k} = \frac{(n-1)!}{k!} \sum_{j=0}^{n-k-1} (-1)^j \frac{n-k-j}{j!} = \binom{n-1}{k} S_{n-k}.$$

(c) $S_n = (n-1)S_{n-1} + (n-2)S_{n-2}$.

4. (15 points; 5,5,5) A finite set of points $S \subset \mathbb{R}^d$ is said to be in **general position** iff no subset of S of cardinality $d+1$ lies in a common $(d-1)$-plane.[1] Let P_d denote any polytope in $\mathbb{R}^d$, with exactly $d+1$ vertices, whose set of vertices is in general position.

   (a) Please calculate the number of k-faces of P_d.

   (b) Please compute the zeta polynomial of the face lattice of P_d.

   (c) Please compute the Mobius function of the face lattice of P_d.

[1]For the purposes of this problem, a k-plane may or may not contain the origin.