Qualifying Exam, January 2003, Number Theory

Instructions and comments

1) Please write your name on each page of the exam paper.

2) There are 4 Number Theory questions worth a total of 50 points (2 questions worth 10 points and 2 questions worth 15 points). Your total grade for the Number Theory section of this exam will be out of a maximum possible of 50 points.

3) The points awarded for your answers will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning.

4) Please write your answers in the space provided in the exam paper.

5) You may use a calculator to check your computations (but you will not earn points for using it as a step in your reasoning).

6) If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

Number Theory, Question 1: total of 10 points

(a) (**5 points**) Let a, b and $m > 0$ be integers. Let $g = (a, m)$, the greatest common divisor of a and m. Show that a necessary and sufficient condition for the congruence

$$ax \equiv b \pmod{m}$$

to have a solution is that g divides b.

(b) (**5 points**) Let p be a prime and c be an integer coprime to p. Show that a necessary and sufficient condition for the congruence

$$y^n \equiv c \pmod{p}$$

to have a solution is that

$$c^{(p-1)/(n,p-1)} \equiv 1 \pmod{p}.$$

Solution to Question 1:

Solution to Number Theory, Question 1, continued:

Number Theory, Question 2: total of 10 points

(a) (**5 points**) Show Wilson's Theorem, namely, that if p is a prime, then

$$(p-1)! \equiv -1 \pmod{p}.$$

(b) (**5 points**) Assume that ξ is a real number with purely periodic continued fraction expansion $< \overline{a_0, a_1, \dots, a_{n-1}} >$, with a_i, $i = 0, \dots n-1$, positive integers. Show that $\xi > 1$ and $-1 < \xi' < 0$, where ξ' denotes the (Galois) conjugate of ξ.

Solution to Question 2:

Solution to Number Theory, Question 2, continued:

Number Theory, Question 3: total of 15 points.

(a) (**7 points**) Let φ be Euler's phi-function. For x a positive real number, let $\pi(x)$ be the number of primes not exceeding x. Show that for any positive integer k, we have the inequality,

$$\frac{\pi(x)}{x} \leq \frac{\varphi(k)}{k} + \frac{2k}{x}.$$

(b) (**4 points**) Let φ be Euler's phi-function and μ the Möbius function. Show that

$$\frac{n}{\varphi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}.$$

(c) (**4 points**) Show that if g and g' are primitive roots modulo an odd prime p, then gg' is not a primitive root mod p.

Solution to Question 3:

Solution to Number Theory, Question 3, continued:

Solution to Number Theory, Question 3, continued:

Number Theory, Question 4: total of 15 points.

(a) (**7 points**) Let p be an odd prime with $p > 3$ and let $(\frac{\cdot}{p})$ be the Legendre symbol. Evaluate the following sum (notice that you are asked to evaluate mod p):

$$\frac{1}{2}\sum_{a=1}^{p-1} a\left(\left(\frac{a}{p}\right)+1\right) \qquad (\operatorname{mod} p).$$

(b) (**8 points**) Let p be an odd prime and n an integer coprime to p. Suppose that n has exactly r distinct prime factors. Let $D(n)$ denote the number of divisors of n that are square-free and that are also quadratic residues mod p. Show that $D(n) = 2^r$ if all the prime factors of n are quadratic residues mod p and that $D(n) = 2^{r-1}$ otherwise.

Solution to Question 4:

Solution to Number Theory, Question 4, continued:

Solution to Number Theory, Question 4, continued: