Qualifying Exam, January 2005, Combinatorics

Instruction and Comments

1. Please write your name on each page of the exam paper.
2. There are 3 Combinatorics questions worth a total of 50 points. Your total grade for the Combinatorics section of this exam will be out of a maximum possible of 50 points.
3. The points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning.
4. Please write your answers on the separated sheets of paper provided.
5. You may use a calculator to check your computations.
6. If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

Question 1. total of 16 points

Let a_n be the number of ways to go up n stairs, where one is allowed to take one or two steps at each time. Set $a_0 = 1$. Find a recurrence for a_n. Then determine the generating function

$$A(x) = \sum_{n \geq 0} a_n x^n,$$

and solve for a_n, (i.e., find an explicit formula for a_n.)

Question 2. total of 16 points

Evaluation the Möbius function $\mu(x, y)$ for the following posets.

1. The lattice of all subsets of a finite set S, ordered by inclusion.
2. A poset P whose Hasse diagram is a rooted tree, where $x \leq y$ if the unique path going from the root to y passes through x.
3. The lattice of all partitions of the set $[n]$, ordered by refinement.

(Note: Proof is not required for this problem. However, in the case that your answer is not correct, some reasoning will help you get partial credit.)

Question 3. total of 18 points

Let $(x)^i := x(x+1)(x+2)\cdots(x+i-1)$, and $(x)_i := x(x-1)(x-2)\cdots(x-i+1)$. Prove the following identities without using induction.

1. $$\frac{(x)^n}{n!} = \sum_{i=1}^{n} \binom{n-1}{i-1} \frac{(x)_i}{i!},$$

2. $$\frac{(x)_n}{n!} = \sum_{i=1}^{n} (-1)^{n-i} \binom{n-1}{i-1} \frac{(x)^i}{i!}.$$