Qualifying Exam, January 2005, Number Theory

Instructions and comments

1) There are 4 Number Theory questions: 2 questions worth 10 points and 2 questions worth 15 points. Your total grade for the Number Theory section of this exam will be out of a maximum possible of 50 points.

2) The points awarded for your answers will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. "Rough working" will not be accepted: answers must be written in a structured and understandable manner.

3) Please write your answers on the separate sheets of paper provided.
Please write your name on each page of answers that you hand in.

4) You may use a calculator to check your computations (but you will not earn points for using it as a step in your reasoning).

5) If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

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The exam questions:

Question 1 (10 points): Let $n = pq$ be a positive integer which is a product of two distinct prime numbers p and q. Show that every integer a satisfies the congruence:

$$a^{pq-p-q+2} \equiv a, \qquad (\mathrm{mod}\ n).$$

Question 2 (10 points): Let ξ be a real quadratic number with purely periodic continued fraction expansion. Show that $\xi > 1$ and $-1 < \xi' < 0$ where ξ' denotes the algebraic (Galois) conjugate of ξ.

Question 3 (15 points): Let p be an odd prime. Let $N(p)$ denote the number of pairs of consecutive quadratic residues mod p in the closed interval $[1, p-1]$. Show that,

$$N(p) = \frac{1}{4}\left(p - 4 - (-1)^{(p-1)/2}\right).$$

Question 4 (15 points): Show that the class number of -20 equals 2: i.e. $H(-20) = 2$. Describe the prime numbers represented by binary quadratic forms of discriminant -20.