Combinatorics Qualifying Exam
January 2006

1. (a) State the Principle of Inclusion-Exclusion.
   (b) Let $1 \le i \le n$. Find the number of permutations of $[n]$ in which i immediately precedes $i+1$. For instance, for $n = 4$ and $i = 2$ these permutations are
   $$1234,\ 4231,\ 2314,\ 2341,\ 1423,\ 4123\,.$$
   (c) Let $1 \le i < j < n$. Find the number of permutations of $[n]$ in which i immediately precedes $i+1$ and j immediately precedes $j+1$. Hint: consider the cases $j = i+1$ and $j > i+1$ separately.
   (d) Let $1 \le i_1 < i_2 < \cdots < i_k < n$. Find the number of permutations of $[n]$ in which i_r immediately precedes $i_r + 1$ for every $r = 1, \ldots, k$.
   (e) Show that the number of permutations of $[n]$ in which no entry i immediately precedes $i+1$ is
   $$\sum_{k=0}^{n} (-1)^k \binom{n-1}{k} (n-k)!\,.$$

2. (a) Write down a definition of the Möbius function of a poset.
   (b) Give explicit formulas for the Möbius function of the Boolean poset (the poset of subsets of $[n]$) and the poset of divisors of n.
   (c) State one form of the Principle of Möbius inversion.

3. Let $S(n,k)$ be the number of partitions of $[n]$ into k blocks (the Stirling number of the second kind).
   (a) Show that $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ (give a combinatorial argument).
   (b) Let $S_k(x) = \sum_{n \ge 0} S(n,k)x^n$. Deduce that $S_k(x) = \dfrac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$.