Qualifying Exam, January 2006, Graph Theory

January 11, 2006

Instruction and Comments

Please write your name on each page of the exam paper.

There are 5 graph theory questions, one having 9 parts. These 5 questions are worth a total of 150 points. After grading, the points earned will be added together and divided by 3, to give a score out of 50. This score will be added to the score earned on the other part of this qualifier to get your total score out of 100.

The points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. Many problems ask for a brief answer ("yes" or "no" or a number) and a proof that that answer is correct. The proof carries most of the credit for the problem.

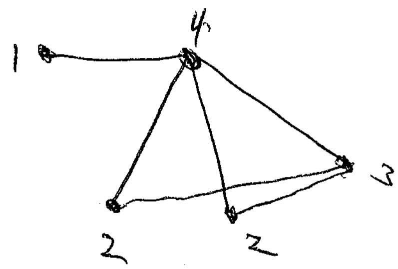
Please write your answers in the space provided in the exam paper. Continue on the back of another page if necessary. Indicate where you continue, and label the continuation with the problem number. The fact that you are given a full page to answer each problem does not mean any particular problem requires that much space.

If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

All of the problems on this exam refer to a graph G_n defined as a graph with n vertices and having vertices of all degrees from 1 through $n-1$. Some problems refer additionally to the graph G'_n formed from G_n by erasing the vertex of degree one.

For Problems 1 through 3, assume G_n is unique for each integer n. In Problem 4, you are asked to prove uniqueness.

The graph G_5 is shown in the following figure, with the vertex degrees labelled.



1) Draw G_1 through G_8, omitting G_5. **(10 points)**

2a) Define: Graph G is a tree. **(4 points)**

2b) For what values of n is G_n a tree? **(3 points)**

3a) Define: Graph G is planar. **(5 points)**

3b) For what values of n is G_n planar? Prove it. **(12 points)**

4) There is a degree x such that two vertices of G_n has two vertices of degree x. What is x? Prove it. Simultaneously, prove that, for each integer n, there is only one graph G_n. **(25 points)**

5) What is the largest value of an integer s such that K_s is a subgraph of G_n? Prove it. **(12 points)**

6) Give a non-trivial description of the complement $\overline{G_n}$ of G_n. (The trivial description is that $\overline{G_n}$ is the complement of G_n.) **(10 points)**

7a) What is the average degree of G_n? (No proof is needed.) **(4 points)**

7b) What is the highest connectivity of a subgraph H of G_n that you can prove exists? Prove it. **(6 points)**

8) A set of vertices in a graph is *independent* iff no two of the vertices in the set are joined by an edge. The independence number $\alpha(G)$ is the size of a largest independent set in G. A *clique* in a graph G is a complete subgraph in G. The clique number $\omega(G)$ is the order of a largest clique in G.

What are the independence number $\alpha(G_n)$ and the clique number $\omega(G_n)$ of G_n? Prove it. **(12 points)**

9a) Define the chromatic number $\chi(G)$ of a graph G. **(6 points)**

9b) What is the chromatic number $\chi(G_n)$? Prove it. **(20 points)**

10a) Is G_n perfect? Circle one of the following. **(1 point)**

YES

NO

10b) This part is a bonus problem and does not have to be done to achieve a perfect score. Prove your answer to problem 10a). **(10 points bonus)**

11a) Define: Graph G is Hamiltonian. **(6 points)**

11b) Form G'_n from G_n by erasing the vertex of degree one. Is G'_n Hamiltonian? Prove it. **(12 points)**

BONUS: This is a bonus problem and does not have to be done to achieve a perfect score.

What is the smallest value of k such that you can prove G'_n has a k-flow? Prove it. **(8 points)**