Qualifying Exam, January 2006, Number Theory

Instructions and Comments

1. Please write your name on the separate sheets provided. Please write your name on each page of answers that you hand in.

2. There are 3 Number Theory questions worth a total of 50 points (1 question worth 20 points, and 2 questions worth 15 points). Your total grade for the Number Theory section of this exam will be out of a maximum possible of 50 points.

3. The points awarded for your answers will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. "Rough working" will not be accepted: answers must be written in a structured and understandable manner.

4. You may use a calculator to check your computations (but you will not earn points for using it as a step in your reasoning).

5. If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

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Exam Questions

1. [15 points] Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of positive integers.

   (a) Let $a : \mathbb{N} \to \mathbb{C}$ and $b : \mathbb{N} \to \mathbb{C}$ be functions. Suppose that we have the identity of formal power series

   $$\sum_{n=1}^{\infty} a(n) \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} b(n) x^n.$$

   Show that

   $$b(n) = \sum_{d|n} a(d).$$

   Is the converse true?

   (b) Define $c : \mathbb{N} \to \mathbb{C}$ by

   $$\sum_{n=1}^{\infty} c(n) x^n = \sum_{n=1}^{\infty} \frac{n x^n}{1-x^n}.$$

   What common arithmetic function is $c(n)$?

   (c) Let $\phi : \mathbb{N} \to \mathbb{C}$ denote Euler's ϕ-function. Consider the formal power series

   $$F(x) = \sum_{n=1}^{\infty} \phi(n) \frac{x^n}{1-x^n}.$$

   Find a polynomial $P(x)$ and an integer k so that the following identity of formal power series holds:

   $$F(x) = \frac{P(x)}{(1-x)^k}.$$

2. [15 points] Let p be an odd prime and fix an integer $k \geq 0$. Derive the following congruence modulo p:

   $$\sum_{a=1}^{p-1} a^k \equiv \begin{cases} -1 \pmod{p} & \text{if } (p-1) \mid k, \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

   Hint: primitive root.

3. [20 points] For an odd prime p and an integer a, let $\left(\frac{a}{p}\right)$ denote the Legendre symbol.

   (a) Assuming $p \geq 5$, fill in the blanks in the formula below correctly and prove your findings:

   $$\left(\frac{-3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \underline{\quad} \pmod{\underline{\quad}} \\ -1 & \text{if } p \equiv \underline{\quad} \pmod{\underline{\quad}} \end{cases}$$

   (b) Prove for any $n \in \mathbb{Z}$ that $n^2 + n + 1$ has no prime divisors of the form $6k - 1$.

   (c) Prove that there are infinitely many primes of the form $6k + 1$.