Combinatorics Qualifying Exam
January 2007

1. Let A be a set with n elements.

   (a) Find the number of sequences in which each element of A appears exactly twice.

   (b) Let $\alpha = (a_1, a_2, \ldots)$ and $\beta = (b_1, b_2, \ldots)$ be two sequences as above. Define

   $$\alpha \sim \beta \iff \text{there is a bijection } \sigma : A \to A \text{ such that } \sigma(a_i) = b_i \text{ for every } i.$$

   Show that this is an equivalence relation and find the number of equivalence classes. 10 pts.

2. Let s_n be the number of solutions $(x_1, x_2, x_3, x_4) \in \mathbb{N}^4$ to the equation

   $$x_1 + x_2 + x_3 + x_4 = n$$

   subject to the conditions that x_1 is even, x_2 is a multiple of 5, x_3 is at most 4, and $x_4 = 0$ or 1. Show that $s_n = n + 1$. 10 pts.

3. (a) Let $\binom{n}{j}$ be the binomial coefficient and $B_j(x) = \sum_{n\geq 0} \binom{n}{j} x^n$.

   Prove that $B_j(x) = \dfrac{x^j}{(1-x)^{j+1}}$.

   (b) Let $S(n,k)$ be the number of set partitions into k blocks and $S_k(x) = \sum_{n\geq 0} S(n,k)x^n$.

   i. Give a combinatorial proof of $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

   ii. Deduce that $S_k(x) = \dfrac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$.

   (c) Let $n, k \geq 0$ and $j = \lceil k/2 \rceil$.

   i. Show that $S_k(x) \equiv x^{k-j+1} B_{j-1}(x) \mod 2$.

   ii. Deduce that

   $$S(n,k) \equiv \binom{n - \lfloor k/2 \rfloor - 1}{n-k} \mod 2\,.$$

   25 pts.

4. Find the number of permutations of the multiset $\{1^2, 2^2, \ldots, n^2\}$ in which no two consecutive terms are equal. 15 pts.

5. Given a finite poset P, define two notions of *height*:

   $$h_c(P) := \max\{M \mid \text{ there is a chain } C \text{ with } M \text{ elements}\}\,,$$
   $$h_a(P) := \min\{m \mid \text{ there are } m \text{ disjoint antichains that cover } P\}\,.$$

   (a) Let C be a chain with M elements and $X_1, \ldots, X_m$ disjoint antichains that cover P. Show that if $x, y \in C$ then there exist $i \neq j$ such that $x \in X_i$ and $y \in X_j$.

   (b) Deduce that $h_c(P) \leq h_a(P)$.

   (c) For each $x \in P$, let C_x be a longest chain having x as minimum element. For each $i \geq 1$, let

   $$X_i := \{x \in P \mid \ C_x \text{ has cardinality } i\}\,.$$

   Show that the sets X_i, $i \geq 1$, are disjoint antichains that cover P.

   (d) Deduce that $h_a(P) \leq h_c(P)$.

   (e) Deduce that $h_c(P) = h_a(P)$.

   20 pts.