Combinatorics Qualifying Exam

January 2008

1. Let n be an integer, $n \geq 2$. A complex number q is an n-th *root of unity* if $q^n = 1$. It is *primitive* if in addition $q^i \neq 1$ for all $i = 1, \ldots, n-1$.

   (a) Show that if q is an n-th root of unity then $[n]_q = 0$ and that if q is primitive then $[i]_q \neq 0$ for all $i = 1, \ldots, n-1$.

   (b) Show that if q is a primitive n-th root of unity then $\binom{n}{i}_q = 0$ for all $i = 1, \ldots, n-1$.

   (c) Let A and B be two $k \times k$-matrices such that

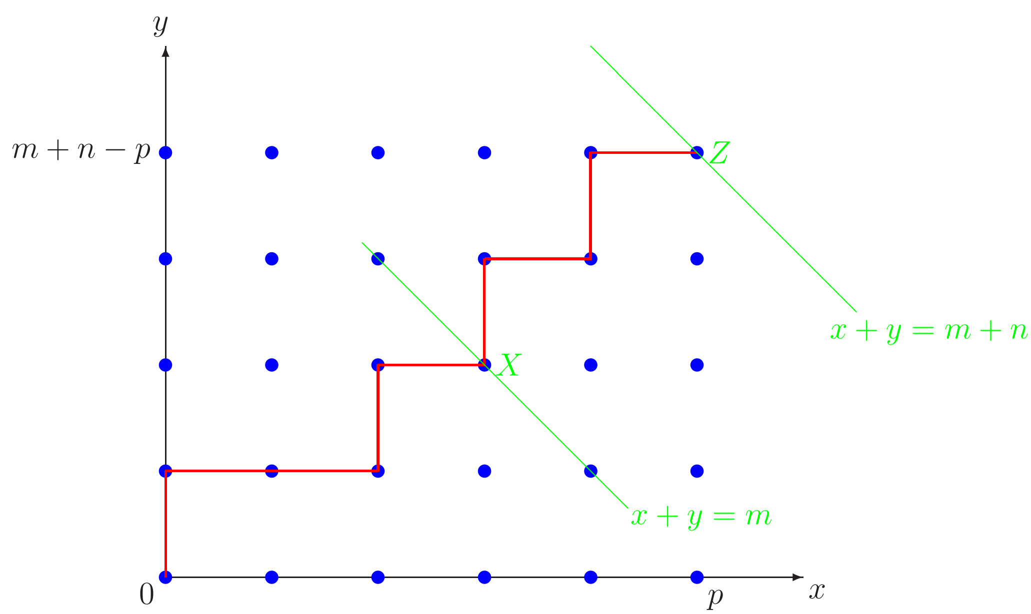
   $$AB = q\,BA.$$

   Show that if q is a primitive n-th root of unity then $(A+B)^n = A^n + B^n$.

2. Vandermonde's formula for the q-binomial coefficients is

   $$\binom{n+m}{p}_q = \sum_{k=0}^{p} q^{(m-k)(p-k)} \binom{m}{k}_q \binom{n}{p-k}_q .$$

   Prove this formula by a lattice path counting argument. The following figure may help.



3. Let $\mathcal{B}_n$ be the Boolean poset of rank n.

   (a) Let $a_1, \ldots, a_k$ be natural numbers such that $a_1 + \cdots + a_k = n$. Find the number of multichains

   $$x_0 \leq x_1 \leq \cdots \leq x_k$$

   in $\mathcal{B}_n$ such that $\mathrm{rk}(x_i) - \mathrm{rk}(x_{i-1}) = a_i$ for all $i = 1, \ldots, k$.

   (b) Find the number of maximal chains in $\mathcal{B}_n$.

   (c) Describe the answers to (a) and (b) for $\mathcal{L}_n(q)$, the poset of subspaces of $\mathbb{F}_q^n$.

4. (a) State the principle of Möbius inversion.

   (b) State and deduce the principle of inclusion-exclusion.

   (c) Find the number of permutations of the multiset $\{1^2, 2^2, \ldots, n^2\}$ in which no two consecutive terms are equal.