Qualifying Exam, May 2004, Combinatorics

Instruction and Comments

1. Please write your name on each page of the exam paper.
2. There are 3 Combinatorics questions worth a total of 50 points. Your total grade for the Combinatorics section of this exam will be out of a maximum possible of 50 points.
3. The points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning.
4. Please write your answers in the space provided in the exam paper.
5. You may use a calculator to check your computations.
6. If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

Combinatorics. Question 1. total of 20 points

1. There are n identical balls and n identical boxes. In each of the following cases, compute the number of different ways that one can place the balls into the boxes.

Question (a). The balls are labeled $1, 2, 3, \ldots, n$, and the boxes are labeled $1, 2, 3, \ldots, n$.

Question (b). The balls are labeled $1, 2, 3, \ldots, n$, and the boxes are labeled $1, 2, 3, \ldots, n$. There is exactly one ball in each box.

Question (c). The balls are labeled $1, 2, 3, \ldots, n$. There is no label on any box. There is exactly one ball in each box.

Question (d). The boxes are labeled $1, 2, 3, \ldots, n$. There is no label on any ball.

Combinatorics. Question 2. total of 15 points

2. We have $2n$ balls in all, with 2 balls labeled i for each $i = 1, 2, 3, \ldots, n$. We also have n boxes labeled $1, 2, 3, \ldots, n$. Let $f(n)$ be the number of ways of placing the balls into the boxes so that

1. Each box contains exactly 2 balls, and
2. no box contains two balls with the same label.

For example, $f(1) = 0$ and $f(2) = 1$.

Question (a). Compute $f(3)$ and list all different ways of placement.

Question (b). State the Principle of Inclusion-Exclusion, (for general n.)

Question (c). Prove that $f(n)$ is given by the formula

$$\frac{\sum_{t=0}^{n}(-1)^t\binom{n}{t}^2 t!\binom{2(n-t)}{2,2,\ldots,2}}{2^n}.$$

(Hint: Let $g(n)$ be the number of ways to list all the balls in a row such that the balls in positions $2i-1$, $2i$ have different labels for each $i = 1, 2, \ldots, n$. What is the relation between $f(n)$ and $g(n)$?)

Combinatorics. Question 3. total of 15 points

3. Recall that the Minkowski sum of 2 subsets A and B of $\mathbb{R}^n$ is the set

$$A + B := \{a + b \mid a \text{ in } A \text{ and } b \text{ in } B\}.$$

For example, in $\mathbb{R}^2$, let $A = \{(x, 0)|0 \leq x \leq 1\}$ and $B = \{(0, y)|0 \leq y \leq 1\}$. Then the Minkowski sum $A + B$ is the unit square $\{(x, y)|0 \leq x, y \leq 1\}$.

Question (a). In $\mathbb{R}^3$, compute the volume of $A + B$ where A is a unit cube and B is a ball of radius $1/2$.

Question (b). Given explicit examples showing that the Minkowski sum of 2 triangles in the plane can have 3, 4, 5, or 6 edges.

Combinatorics, Extra Paper

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