Qualifying Exam, May 2004, Graph Theory

May 28, 2004

Instruction and Comments

Please write your name on each page of the exam paper.

There are 5 graph theory questions, one having 8 parts. These 5 questions are worth a total of 135 points. After grading, the points earned will be added together and converted to a grade out of 50. This score will be added to the score earned on the other part of this qualifier to get your total score out of 100.

The solution to Problem 3(f) is important for solving later questions. If you cannot solve 3(f), you may obtain the solution from the proctor at the cost of a 0 on 3(f).

The points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning.

Please write your answers in the space provided in the exam paper. Continue on the back of another page if necessary. Indicate where you continue, and label the continuation with the problem number.

If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

1) State Seymour's 6-flow theorem. **(5 points)**

2)(a) Define the Ramsey number $R(n)$. **(5 points)**

2)(b) State Ramsey's Theorem (9.1.1). **(5 points)**

3) Let d be a positive integer, and let

$$V_d = \{\text{all 0,1 sequences of length } d\}.$$

Form graph Q_d on vertex set V_d by joining each element x of V_d with each of the elements of V_d which differs from x in exactly one place. For example, Q_2 is shown here.

DRAWING SUPPLIED AT TIME OF TEST.

3)(a) Draw Q_1 and Q_3. **(2 + 4 points)**

3)(b) Prove that Q_d is bipartite for every integer $d \geq 1$. **(12 points)**

3)(c) Without referencing a theorem, explain why Q_{18} has a nwz-Z_2-flow. **(8 points)**

3)(d) State Euler's formula and use it to prove that Q_4 is not planar. **(10 points)**

3)(e) State the edge-coloring number $\chi'(Q_d)$ and name the theorem used to find it. **(5 points)**

3)(f) Develop a construction that produces Q_{d+1} from Q_d. This construction is important for later questions. **(12 points)**

3)(g) A *1-factor* of a graph G is a matching which meets every vertex of G. Prove that Q_d has a 1-factor for every $d \geq 1$ and give the minimum cardinality of a vertex cover of Q_d. **(10+2 points)**

3)(h) Prove that, for every integer $d \geq 2$, the graph Q_d is Hamiltonian. **(10 points)**

4)(a) State the max-flow, min-cut theorem. **(5 points)**

4)(b) Use the max-flow, min-cut theorem to prove the following theorem:

Let G be a graph with distinct vertices a and b. Then the minimum number of edges separating a from b in G is equal to the maximum number of edge-disjoint a, b-paths in G. **(12 points)**

5) Let $P(G, \lambda)$ be the number of ways graph G can be vertex- colored with λ colors.
5)(a) Prove: If $e \in E(G)$, then $P(G, \lambda) = P(G - e, \lambda) - P(G/e, \lambda)$. **(16 points)**

5)(b) Prove that $P(G, \lambda)$ is a polynomial of degree $n = |V(G)|$. **(12 points)**