Qualifying Exam, May 2005, Combinatorics

Instruction and Comments

1. Please write your name on each page of the exam paper.
2. There are 3 Combinatorics questions worth a total of 50 points. Your total grade for the Combinatorics section of this exam will be out of a maximum possible of 50 points.
3. The points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning.
4. Please write your answers on the separated sheets of paper provided.
5. You may use a calculator to check your computations.
6. If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

Question 1. total of 16 points

Let $|S| = n$, and fix a positive integer k. How many sequences $(T_1, T_2, \ldots, T_k)$ of subsets T_i of S are there such that $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$? Explain how you get your answer.

Question 2. total of 16 points

Evaluation the Möbius function $\mu(x, y)$ for the following posets.

1. The lattice of all subsets of a finite set S, ordered by inclusion.

2. The lattice of all partitions of the set $[n]$, ordered by refinement.

3. $\mathbb{Z} \times \mathbb{Z}$, the direct product of $\mathbb{Z}$ with itself, i.e., $(x, y) \le (x'.y')$ iff $x \le x'$ and $y \le y'$.

(Note: Proof is not required for this problem. However, in the case that your answer is not correct, some reasoning will help you get partial credit.)

Question 3. total of 18 points

Define $S(n, k)$ to be the number of partitions of an n-set into k blocks. $S(n, k)$ is called a *Stirling number of the second kind.* By convention, we put $S(0, 0) = 1$. It is easy to check that $S(n, 0) = 0$ for $n \ge 0$, $S(n, 1) = 1$, $S(n, 2) = 2^{n-1} - 1$, and $S(n, n) = 1$. (You don't need to check them.)

Prove the following identities about $S(n, k)$.

1.
$$S(n, k) = kS(n-1, k) + S(n-1, k-1).$$

2.
$$x^n = \sum_{k=0}^{n} S(n, k)(x)(x-1)\cdots(x-k+1).$$

3.
$$\sum_{n \ge k} S(n, k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k.$$