Qualifying Exam, May 2005, Number Theory

Instructions and comments

1) Please write your name on the separate sheets provided.
Please write your name on each page of answers that you hand in.

2) There are 3 Number Theory questions worth a total of 50 points (2 questions worth 15 points and 1 questions worth 20 points). Your total grade for the Number Theory section of this exam will be out of a maximum possible of 50 points.

3) The points awarded for your answers will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning. “Rough working” will not be accepted: answers must be written in a structured and understandable manner.

4) You may use a calculator to check your computations (but you will not earn points for using it as a step in your reasoning).

5) If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

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The exam questions

Question 1 (15 points)

(1)(a) Let m be a positive integer that is not a power of 2. If m has a primitive root, show that $m = p^k$ or $2p^k$, where p is an odd prime and k is a positive integer.

(1)(b) Show that if $m = 2^k$, there exists a primitive root mod m if and only if $m = 2$ or 4.

Question 2 (15 points) Show that, for $s > 1$, we have

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s},$$

where $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \prod_{p,\text{prime}} \left(1 - p^{-s}\right)^{-1},$$

and $\mu(n)$ is the Mobius function.

Question 3 (20 points)

Let d be a positive integer which is not a perfect square. Let x_1, y_1 be the least positive solution of

$$x^2 - dy^2 = 1.$$

Show that *ALL* positive solutions of this equation are given by integers x_n, y_n defined by

$$x_n + y_n\sqrt{d} = \left(x_1 + y_1\sqrt{d}\right)^n, \quad n = 1, 2, 3, \dots .$$