Combinatorics Qualifying Exam

May 2006

1. (a) Find the number of lattice paths from $(0,0)$ to $(9,9)$ that touch the point $(3,3)$ but do not touch the point $(6,6)$.

   (b) Find the number of lattice paths as in part (i) that in addition never go below the diagonal.

2. Let $S(n,k)$ be the number of partitions of $[n]$ into k blocks (the Stirling number of the second kind).

   (a) Show that

$$S(n,2) = \begin{cases} \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{\frac{n}{2}-1} + \frac{1}{2}\binom{n}{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

   (b) Deduce that $S(n,n-1) < S(n,2)$ for every $n > 3$..

3. Let $A(x)$ be the generating function of a sequence $a_0, a_1, a_2, \ldots$. Suppose you know that $a_0 = 1$ and

$$A(x)^2 = 2xA(x) + 1.$$

   (a) Find $A(x)$.

   (b) Find a_n.

4. You are given n rectangular pieces, each 3 cm $\times$ 1 cm, and each with a top black face and a bottom yellow face. Let a_n be the number of ways of arranging the pieces with no overlapping on a tray of dimensions 3 cm $\times$ n cm. For instance, one arrangement for $n = 5$ is

   Assume $a_0 = 1$.

   (a) Find a recursion for a_n.

   (b) Find a generating function for a_n.

   (c) Find a_n.

5. (a) Let P be a finite poset and μ its Möbius function. Define $\alpha : P \to \mathbb{N}$ by

$$\alpha(y) = \#\{x \in P \mid x \leq y\}.$$

   Show that for each fixed $y \in P$,

$$\sum_{x \leq y} \mu(x,y)\alpha(x) = 1.$$

   (b) Let G_n be the number of subspaces of a vector space of dimension n over the field $\mathbb{F}_q$. Deduce that

$$\sum_{k=0}^{n} \binom{n}{k}_q (-1)^k q^{\binom{k}{2}} G_{n-k} = 1.$$