

Qualifying Exam, May 2006, Graph Theory

May 18, 2006

Instruction and Comments

Please write your name on each page of the exam paper.

There are 7 graph theory questions, the first having six parts and the third having three parts. These 7 questions are worth a total of 150 points. After grading, the points earned will be added together and divided by 3, to give a score out of 50. This score will be added to the score earned on the other part of your qualifier to get your total score out of 100.

The points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning.

Please write your answers in the space provided in the exam paper. Continue on the back of another page if necessary. Indicate where you continue, and label the continuation with the problem number.

If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

1a) (5 points) State Brooks' coloring theorem (~~5.2.4, p. 99 of Diestel's book~~).

1b) (5 points) State the definition of a well-quasi-ordering (~~pp. 251-2 of Diestel~~).

1c) (5 points) State the Robertson and Seymour graph minor theorem (~~12.5.2 of Diestel~~).

1d) (5 points) State Erdős' girth and chromatic number theorem.

1e) (5 points) State the 5-color theorem.

1f) (5 points) Suppose graph G has a subgraph H isomorphic to K^{48} . Suppose further that every subgraph of G has a vertex of degree at most 53. Give bounds for the chromatic number $\chi(G)$.

2) (10 points) Prove that, if G is a 3-connected simple graph, then K^4 minus an edge is a minor of G .

3a) (5 points) Define the Ramsey number $R(k)$.

3b) (5 points) State Ramsey's Theorem (~~8.1.1~~).

3c) (18 points) Prove Erdős' lower bound, that $R(k) > 2^{k/2}$.

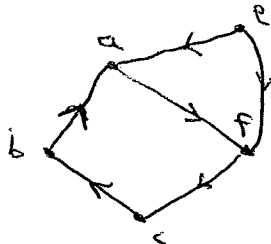
4) (15 points) A graph G is *hypohamiltonian* if it is not Hamiltonian but $G - x$ is Hamiltonian for every vertex x of G . Prove every hypohamiltonian graph with at least four vertices is connected.

5) (23 points) Prove: If G is a connected graph and P and Q are maximum length paths in G , then $V(P) \cap V(Q) \neq \emptyset$.

6) (19 points) A simple graph is *self-complementary* if it is isomorphic to its complement. Prove that a self-complementary graph has either $4k$ or $4k + 1$ vertices for some integer k .

7) (25 points) We orient an edge e of an undirected graph by naming one end of e the **head** of e and the other end of e the **tail** of e . We speak of the edge as being “oriented from its tail to its head.” An oriented edge with tail v and head w is denoted by (v, w) . A graph is **oriented** or **directed** if and only if every edge of the graph is oriented, and it is said to be a **digraph**.

A **dicycle** is any sequence $v_1, v_2, \dots, v_k, v_{k+1} = v_1$ of otherwise distinct vertices such that (v_i, v_{i+1}) is an oriented edge of the digraph for each $i \in \{1, 2, \dots, k\}$. For example, in the following digraph, a, f, c, b, a is a dicycle, and e, a, f, e is not a dicycle.



Prove: If D is a digraph with no dicycles, then D has a vertex v such that every edge of D incident with v has v as a tail. (That is, every such edge is directed away from v .)