Qualifying Exam, May 2007, Graph Theory

May 22, 2007

Instruction and Comments

Please write your name on each page of the exam paper.

There are 9 graph theory questions, the first having 5 parts, and the third and fifth having 3 parts each. These 9 questions are worth a total of 150 points. After grading, the points earned will be added together and divided by 3, to give a score out of 50. This score will be added to the score earned on the other part of your qualifier to get your total score out of 100.

The points awarded for your answer will be based on the correctness of your answer as well as the clarity of the main steps in your reasoning.

Please write your answers in the space provided in the exam paper. For most problems, there should be more than enough space; we've tried to be generous with paper. If there is not enough room after the statement of the problem, continue on the back of another page. Indicate where you continue, and label the continuation with the problem number.

If you finish early, please hand your paper in to the person proctoring the exam and leave the room quietly.

1a) **(5 points)** State König's theorem on matchings (2.1.1, p. 35 of Diestel's book).

1b) **(5 points)** State Tutte's theorem on 3-connected graphs (3.2.2, p. 58 of Diestel's book).

1c) **(5 points)** State Euler's formula (4.2.9, p. 91 of Diestel's book).

1d) **(5 points)** State the definition of a circulation (pp. 140-141 of Diestel's book).

1e) **(5 points)** State the independence number theorem (10.1.2, p. 277 of Diestel's book).

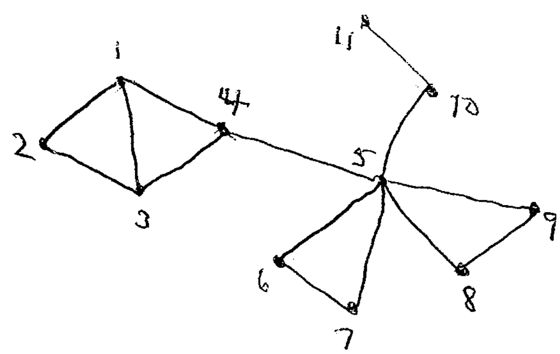
2) **(18 points)** Prove Hall's Theorem: Suppose graph G is bipartite with bipartition (A, B). Then G contains a matching of A if and only if $|N(S)| \geq |S|$ for all $S \subseteq A$.

3) Let G be the graph drawn on this page.

3a) **(8 points)** Draw the blocks of G. as individual graphs.

3b) **(8 points)** Draw the block–cut-vertex tree of G.

3c) **(6 points)** Draw the dual of G.



4) **(12 points)** Suppose graph G is regular of degree $k \geq 2$, and suppose G is bipartite. Prove G is bridgeless.

5a) **(6 points)** A set of vertices in a graph is *independent* iff no two of the vertices in the set are joined by an edge. Suppose a graph G with $2n - 1$ vertices has no three independent vertices. Prove that, if G is not connected, then G has a subgraph isomorphic to K^n.

5b) **(12 points)** Find the largest integer a and an integer $b \geq n$ such that you can prove that, if G has $2n - 1$ vertices, has no three independent vertices, and has a vertex of degree a, then G has a subgraph isomorphic to K^b. Prove it.

5c) **(3 points)** Explain why your prior results show that a graph of $2n-1$ vertices and no three independent vertices has minimum degree at least $a + 1$ if it has no K^b-minor.

6) **(12 points)** Prove that, in a 2-connected graph G, any two longest cycles share at least two vertices.

7) **(8 points)** Find the properties of a graph G having chromatic polynomial

$$x^6 - 5x^5 + 9x^4 - 7x^3 + 2x^2.$$

8) **(12 points)** Suppose G is a planar graph with $|G| \geq 4$. Suppose further that there is a plane embedding of G in which every face has at least five edges on its boundary, bridges counted twice. Find a bound on $|E(G)|$ in terms of $|V(G)|$.

9) **(20 points)** Suppose G is a connected graph with average degree equal to 17.6. Prove that G includes a path of length 10.