Number Theory Qualifier Exam
May 23, 2007
1:00-5:00

Instructions:

- There are three Number Theory questions worth a total of 50 points. Individual point values are listed next to each problem.
- Credit awarded for your answers will be based on the correctness of your answer as well as the clarity and main steps of your reasoning. "Rough working" will not be accepted: answers must be written in a structured and understandable manner.
- You may use a calculator to check your computations (but you will not earn points for using it as a step in your reasoning).

Problems:

1. [15 points] Consider the sequence of rational numbers $\{t_n\}_{n\geq 1}$,

$$t_1 = 1\tfrac{1}{3},\ t_2 = 2\tfrac{2}{5},\ t_3 = 3\tfrac{3}{7},\ t_4 = 4\tfrac{4}{9},\ t_5 = 5\tfrac{5}{11}, \ldots,$$

where the general term follows the obvious pattern.

   (a) For each n, we can express t_n as a fraction in lowest terms. Show that the numerator and denominator of t_n are members of a primitive Pythagorean triple (a, b, c). (Primitive in the sense that a, b, and c are collectively relatively prime.)

   (b) Suppose (a, b, c) is a primitive Pythagorean triple with $c = b + 1$. Show that (a, b, c) is generated by the method in part (a).

   (c) Is the number of primitive Pythagorean triples that do not arise in this way finite or infinite?

2. [15 points] Let p and q be odd prime numbers.

   (a) Show that if p divides $2^q - 1$ then $p \equiv 1 \pmod q$ and $p \equiv \pm 1 \pmod 8$.

   (b) Show that if $p \equiv 5 \pmod{12}$ then p divides $3^{(p-1)/2} + 1$.

3. [20 points] Let n be a positive integer. Let ϕ denote Euler's ϕ-function, and let μ denote the Möbius function. For a real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Derive the following identities.

   (a) $\displaystyle \frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$

   (b) $\displaystyle \sum_{k=1}^{n} \mu(k) \left\lfloor \frac{n}{k} \right\rfloor = 1$

   (c) $\displaystyle \sum_{k=1}^{n} \frac{\phi(k)}{k} = \sum_{k=1}^{n} \frac{\mu(k)}{k} \left\lfloor \frac{n}{k} \right\rfloor$

   (d) $\displaystyle \sum_{k=1}^{n} \phi(k) = \frac{1}{2}\left(1 + \sum_{k=1}^{n} \mu(k) \left\lfloor \frac{n}{k} \right\rfloor^2\right)$