

Complex Analysis Qualifying Exam

January, 2000

Start each problem on a different sheet of paper. Write only on one side of the page. No calculators, books or notes are allowed. You may use any standard theorem in complex analysis except the ones you are asked to prove.

1. Compute $\int_0^{\infty} \frac{\ln x}{1+x^4} dx$.
2. Find a biholomorphic map which takes the set $\{|z - \sqrt{3}| < 2\} \cap \{\operatorname{Re} z < 0\}$ to the half-infinite strip $\{0 < \operatorname{Re} z < 1\} \cap \{\operatorname{Im} z > 0\}$.
3. Suppose $\sum_{n=-\infty}^{\infty} a_n z^n$ is the Laurent series of $\frac{\cot z}{z(z-2)}$ valid on the region $\{2 < |z| < \pi\}$. Find a_n for $n < 0$.
Hint: The computation of a_n for $n < 0$ is much easier than for $n > 0$.
4. (a) Prove the *Schwarz Lemma*: if $f : \Delta \mapsto \Delta$ is analytic (Δ is the open unit disc centered at the origin) and $f(0) = 0$, then $|f(z)| \leq |z|$ for all $z \in \Delta$ and $|f'(0)| \leq 1$.
(b) Let $U = \{\operatorname{Im} z > 0\}$ and suppose $f : U \mapsto U$ is holomorphic with $f(i) = i$. Show that $|f'(i)| \leq 1$.
5. Suppose D is an open set and suppose f_n is a sequence of holomorphic functions on D which converges uniformly on each compact subset of D to a non-constant function f . Let z_0 be a point in D . Show that there is a sequence $z_n \in D$ with $z_n \mapsto z_0$ with $f_n(z_n) = f(z_0)$ for all sufficiently large n .
6. Show that the set of biholomorphic maps from C to C is the set

$$\{f(z) = az + b; a \text{ and } b \text{ are complex numbers with } a \neq 0\}.$$

7. Let Δ be the open unit disc centered at the origin. Show that the set

$$\left\{ f : \Delta \rightarrow \mathbb{C} : f \text{ is holomorphic and } \int_{z+iy \in \Delta} |f(x+iy)| dx dy < 1 \right\}$$

is a normal family.

8. (a) Let $E = \{|z| = 1 \text{ and } \operatorname{Im} z \geq 0\}$. Does there exist a sequence of (holomorphic) polynomials in z which converge uniformly on E to $f(z) = \bar{z}$? Justify your answer.
(b) Answer the same question in part (a) for the set $E = \{|z| = 1\}$.
9. Find the infinite product representation of an entire function with simple roots at the points \sqrt{n} , $n = 1, 2, \dots$ and nonvanishing everywhere else. Provide justification for your answer and carefully state any theorems that you use.
10. Carefully state the Riemann Mapping Theorem. Outline the key steps in its proof.