# Complex Analysis Qualifying Exam, January 2002.

(each problem is worth 10 points.)

1. Let $\{f_n\}_{n=1}^{\infty}$ be a uniformly bounded sequence of nonconstant holomorphic functions in a connected open set $\Omega$. Let $f \not\equiv 0$ be a holomorphic function in $\Omega$. Suppose that the equation

$$nf(z) = f_n(z) + \ln n$$

does not have any roots in $\Omega$ for $n = 1, 2, ....$ Prove that then $f$ does not have any zeros in $\Omega$.

2. Can the function defined as $e^{-1/x}$ on $[1, \infty)$ be extended to a function analytic in the whole complex plane?

3. Calculate

$$\int_0^{\infty} \frac{dx}{1 + x^n},$$

where $n \in \mathbb{N}$, $n \geq 2$, using residues.

4. Find the general formula for a conformal map that sends the unit disk $\{|z| < 1\}$ onto the half-plane with a slit $\{Rez > 0\} \setminus [0, 1]$. (You need to show that no other maps exist.)

5. State Runge's and Mergelyan's Theorems (approximation by rational functions and by polynomials respectively).

6. Let $f$ be a holomorphic function in a bounded open domain $\Omega$. Suppose that $\limsup_{n\to\infty} |f(z_n)| \leq 1$ for any sequence $\{z_n\}_{n=1}^{\infty} \subset \Omega$ that converges to a point in $\partial\Omega$. Prove that then $|f| \leq 1$ in $\Omega$.

7. Let $f_n$ be a sequence of holomorphic functions in a connected open set $\Omega$. Suppose that for some $w_1, w_2 \in \Omega$ the product $\prod_n f_n(w_1)$ diverges to infinity (partial products tend to infinity) and $\prod_n f_n(w_2)$ converges to a non-zero constant. Show that then for infinitely many $n$, $\inf_\Omega |f_n| < 1$. (Hint: apply log and use Harnack's Principle.)

8. Prove that every meromorphic function in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is rational.

9. Let $u$ be a positive harmonic function in $\{Rez > 0\}$ such that

$$\lim_{r\to 0+} u(r) = 0.$$

Prove that then $\lim_{r\to 0+} u(re^{i\phi}) = 0$ for any $\phi \in (-\pi/2, \pi/2)$.

10. Let $f$ be holomorphic in a neighborhood of $\{|z| \leq 1\} \setminus \{1/2\}$ and have a pole of order $n$ at $1/2$. Suppose that $|f| < 3$ on $\{|z| = 1\}$. Show that for any $\phi \in \mathbb{R}$, $f$ attains the value $3e^{i\phi}$ exactly $n$ times (counting multiplicities) in $\{|z| < 1\}$.