Complex Analysis Qualifying Exam

January, 2003

Start each problem on a different sheet of paper. Write only on one side of the page. No calculators, books or notes are allowed. You may use any standard theorem in complex analysis except the ones you are asked to prove.

1. Compute $\displaystyle\int_0^\infty \frac{dx}{x^{1/3}(1+x)}$.

2. Give an explicit description of the set of automorphisms of the punctured complex plane, $C - \{0\}$ (recall that an automorphism of a domain is a biholomorphic map from the domain onto itself).

3. Suppose f is entire and that the restriction of f to the real axis is real-valued and the restriction of f to the imaginary axis is imaginary-valued. Show that f is odd, i.e. $f(-z) = -f(z)$.

4. Find a biholomorphic map which takes the set $\{|z - \sqrt{3}| < 2\} \cap \{\text{Re}z < 0\}$ to the half-infinite strip $\{0 < \text{Re}z < 1\} \cap \{\text{Im}z > 0\}$.

5. Is the following statement true or false: there does NOT exist a non-constant function u which is both harmonic in the entire complex plane and bounded from above (i.e. there is an $M < \infty$ with $u(x, y) \leq M$ for all $z = x + iy \in C$). Either prove this statement or find a counter-example.

6. Find a nonconstant entire function with zeros at $z = \sqrt{n}$ of order n for $n = 1,\ 2, \ldots$.

7. Let $D(0, 1)$ be the open unit disc and $RHP = \{\text{Re}z > 0\}$. Show that the set of functions given by $\{f : D(0, 1) \mapsto RHP;\ f \text{ is holomorphic and } f(0) = 1\}$ is a normal family. Is the assertion true without any condition on $f(0)$?

8. Suppose f_n is a sequence of holomorphic functions on a domain D which converges uniformly on compact subsets of D to a nonconstant function f. Suppose $f(a) = b$ for some $a \in D$. Show that there is a sequence $z_n \in D$ with $z_n \mapsto a$ and such that $f_n(z_n) = b$ for all sufficiently large n.

9. Suppose f is entire and $|f(z)| = 1$ for all $|z| = 1$. Show that $f(z) = Cz^n$ for some non-negative integer n and some constant C with $|C| = 1$.

10. Prove the following statement without quoting Runge's approximation theorem (this is one of the key steps in its proof): suppose K is a compact set in C and let D be the unbounded connected component of $C - K$ (the complement of K); if a belongs to D, show that $f(z) = 1/(z - a)$ can be approximated uniformly on K by a sequence of holomorphic polynomials.