Complex Analysis Qualifying Examination

January 2004

1. Prove that the images of two complex numbers z and w under stereographic projection to the sphere are diametrically opposite points of the sphere if and only if $z \cdot \overline{w} = -1$.

2. Derive the Cauchy-Riemann equations in polar coordinates.

   More precisely, suppose that f is a complex-valued function defined in a neighborhood of a non-zero point p in the complex plane, and suppose that the complex derivative $f'(p)$ exists. Let u and v denote the real and imaginary parts of f, and let r and θ denote the standard polar coordinates. Prove that the first-order partial derivatives of u and v with respect to r and θ exist at p and satisfy the equations

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r} \cdot \dfrac{\partial v}{\partial \theta} \qquad \text{and} \\ \dfrac{1}{r} \cdot \dfrac{\partial u}{\partial \theta} = -\dfrac{\partial v}{\partial r}\,. \end{cases}$$

3. Exhibit a closed path γ in the complex plane such that

$$\frac{1}{2\pi i} \oint_\gamma \frac{1}{(z-1)(z-14)}\, dz = 2004.$$

4. Prove that if b is a positive real number less than 1, then

$$\int_0^\infty \frac{x^b}{(1+x)^2}\, dx = \frac{\pi b}{\sin(\pi b)}.$$

5. Give a concrete example of a holomorphic function on the open unit disc such that the real part of the function is bounded, and the imaginary part of the function is unbounded.

6. Let f be a holomorphic function in a neighborhood of the closed unit disc $\{ z \in \mathbb{C} : |z| \leq 1 \}$, and let u denote the real part of f. Prove the following version of Schwarz's formula that recovers f from u:

$$f(w) = -\overline{f(0)} + \frac{1}{\pi i} \oint_{|z|=1} \frac{u(z)}{z-w}\, dz \qquad \text{when } |w| < 1.$$

7. Let G denote the open unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$ with the origin removed and additionally with the point 2^{-n} removed for every positive integer n. Suppose that f is a holomorphic function on G, and suppose that each deleted point of the form 2^{-n} is a pole of f.

   Since the origin is a non-isolated singularity of f, the standard terminology of "removable/pole/essential" does not apply to the origin. Show, however, that the origin "acts like" an essential singularity in the sense of the Casorati-Weierstrass theorem. Namely, prove that for every positive real number δ, the range of the restriction of f to the domain $G \cap \{ z \in \mathbb{C} : |z| < \delta \}$ is dense in $\mathbb{C}$.

8. Prove that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers, then the following two statements are equivalent.

   (a) The infinite product $\displaystyle\prod_{n=1}^{\infty} \frac{a_n + iz}{a_n - iz}$ converges uniformly on compact subsets of the open upper half-plane $\{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$ to a holomorphic function of z on the open upper half-plane.

   (b) The infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges (to a finite value).

9. (a) Prove that there exists a holomorphic function f on the open unit disc $\{ z \in \mathbb{C} : |z| < 1 \}$ with the properties that $f(0) = 0$ and $f(1 - \frac{1}{n}) = 1$ for every integer n greater than 1.

   (b) Prove that every such holomorphic function is unbounded.

10. State and sketch the proof of *one* of the following three theorems: Hadamard's three-circles theorem (concerning the maximum modulus of a holomorphic function on concentric circles); Runge's theorem (concerning approximation by rational functions); Hadamard's factorization theorem (concerning the order and genus of entire functions).