

Complex Analysis Qualifying Examination
January 2005

All problems are worth 10 points.

1) Find a biholomorphic map from the domain $\{z : \operatorname{Im} z > 0 \text{ and } |z| > 1\}$ onto the upper half plane $\{z : \operatorname{Im} z > 0\}$.

2) Let A be the annulus $\{z : 1 < |z| < 2\}$. Let f be a nonvanishing holomorphic function in A .

a) Show that if $\sum_{n=-\infty}^{\infty} a_n z^n$ is the Laurent expansion of f'/f in A , then a_{-1} is an integer and f has a holomorphic logarithm in A exactly when this integer is 0.

b) Show that there exist an integer n and a holomorphic function g on A such that $f(z) = z^n e^{g(z)}$. Moreover, n is unique.

3) Assume that f is continuous on the closed unit disc \overline{D} and holomorphic on D , with $|f(z)| = 1$ when $|z| = 1$. Show that f is the restriction to \overline{D} of a rational function.

4) Prove Vitali's theorem: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions on the unit disc D which is uniformly bounded on compact subsets of D . Assume that $\lim_{n \rightarrow \infty} f_n(a_j)$ exists for all $j \in \mathbb{N}$, where $\{a_j\}_{j=1}^{\infty}$ is a sequence in D which has an accumulation point in D . Then $\{f_n(z)\}_{n=1}^{\infty}$ converges uniformly on compact subsets of D .

5) Show that there exists no one-to-one holomorphic map of the punctured disc $\{z : 0 < |z| < 1\}$ onto the annulus $\{z : 1 < |z| < 2\}$.

6) Assume that f is holomorphic in the unit disc D and continuous on the closure \overline{D} . Show that if f is one-to-one on the unit circle bD , then it is one-to-one on \overline{D} .

Hint: You may use the topological fact that a Jordan curve winds exactly once around each point of its interior.

7) Prove that the entire function $1 - \sin(z)$ has an entire square root; in other words, there exists an entire function f such that $f(z)^2 = 1 - \sin(z)$ for all z .

8) Prove that the infinite product $\prod_{n=1}^{\infty} (1 + (\frac{z}{n})^n)$ converges for all z and represents an entire function of order 2.

9) For a compact set K in the plane, denote by \widehat{K} the set of points z with $|p(z)| \leq \max_{w \in K} |p(w)|$ for all polynomials $p(z)$. Show that \widehat{K} equals the union of K with all the bounded components of $\mathbb{C} \setminus K$.

10) For *one* of the following, state the theorem and sketch its proof: Morera's theorem, Harnack's principle, the monodromy theorem.