Complex Analysis
Qualifying Examination

January 2007

Each problem is worth 10 points.

1. Suppose m and n are positive integers such that for every non-zero complex number z, the expression $\left(z^{1/n}\right)^m$ represents the same set of complex values as the expression $(z^m)^{1/n}$. Prove that m and n are relatively prime.

2. Suppose that $f(z)$ is a complex-valued function defined in a neighborhood of 0 in $\mathbb{C}$, and $f(0) = 0$. Prove that if f is differentiable at 0 in the real sense, and if the (two-dimensional) limit $\lim_{z\to 0} |f(z)/z|$ exists, then either $f(z)$ or $\overline{f(z)}$ is differentiable at 0 in the complex sense.

3. Use contour integration to show that
$$\int_0^\infty \frac{1}{x^{1/6}(x+64)}\,dx = \pi.$$

4. Find the linear fractional transformation (Möbius transformation) that maps the triangle with vertices 0, 1, and i onto the triangle with vertices 0, 2, and $1+i$.

5. Find the Laurent series (in powers of z and $1/z$) that represents the rational function $\dfrac{z^2-2z+7}{(z-3)(z^2+1)}$ in the annulus where $1 < |z| < 3$.

6. Determine how many zeroes the polynomial $z^4 + 5z + 1$ has inside the unit square where $0 < \operatorname{Re} z < 1$ and $0 < \operatorname{Im} z < 1$.

7. Suppose that f is a holomorphic function on the upper half-plane that maps the half-plane biholomorphically onto a regular hexagon. Prove that f is *not* the restriction of an entire function to the upper half-plane.

8. Find all non-constant entire functions f such that $|f'(z)| \le |f(z)|$ for all z in $\mathbb{C}$.

9. Find all non-constant entire functions f such that $f \circ f \circ f = f$, that is, such that $f(f(f(z))) = f(z)$ for all z in $\mathbb{C}$.

10. State and prove *one* of the following theorems: the Riemann mapping theorem, Runge's approximation theorem, or the Weierstrass factorization theorem.