Complex Analysis Qualifying Exam, January 2008

Problem 1. Let $a > 0$. Compute

$$\int_0^\infty \frac{\sqrt{x}\,dx}{(x^2+a^2)^2}\ .$$

Problem 2. Assume f is a periodic entire function with period 1, i.e. $f(z+1) = f(z)$ for all $z \in \mathbb{C}$. For $w \in \mathbb{C}$, set

$$F(w) := \int_{\gamma_w} f(z)dz\ ,$$

where $\gamma_w(t) = w + 1 + e^{it}$, $-\pi \le t \le 0$. Show that F is constant.

Problem 3. Find all holomorphic functions f in $\mathbb{C} \setminus \{0\}$ that satisfy the estimate

$$|f(z)| \le C\left(\sqrt{|z|} + \frac{1}{\sqrt{|z|}}\right)$$

for some constant $C = C(f)$.

Problem 4. Find the number of zeroes (counting multiplicities) that the function $f(z) = 4z^4 - z - e^z$ has inside the annulus $\{z \in \mathbb{C} \mid \frac{1}{10} < |z| < 1\}$. (Justify your answer!)

Problem 5.

a) Show that there exists a holomorphic branch of $f(z) = z^{1/3}(z+1)^{1/2}$ on $D(4,4)$, the disc of radius 4 centered at 4.

b) Let γ be a closed C^1-curve starting and ending at 4 that winds n times around 0 and m times around -1, respectively. The germ of f at 4 admits analytic continuation along γ (why?). Determine the pairs $(n,m) \in \mathbb{N} \times \mathbb{N}$ with the property that the germ at 4 resulting from this analytic continuation agrees with the germ of f at 4.

Problem 6. Denote by Ω the annulus $\{z \in \mathbb{C} \,|\, 1 < |z| < 2\}$ with the interval $(1, 2)$ removed.

a) Show that if $f : \Omega \to \mathbb{C}$ is holomorphic, then there is a sequence of polynomials in z that converges to f locally uniformly on Ω.

b) If f is bounded, can the sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials be chosen so that $\sup_{z\in\Omega} \sup_{n\in\mathbb{N}} |p_n(z)| < \infty$? (Justify your answer.)

Problem 7. Let Ω consist of the unit disc with the interval $[0, 1)$ removed. Find a one-to-one holomorphic map of Ω onto the unit disc.

Problem 8. Let f be a holomorphic automorphism of a bounded domain Ω. Set $f_1 = f$ and $f_n = f \circ f_{n-1}$ when $n > 1$.

a) Show that there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ that converges uniformly on compact subsets of Ω to a holomorphic function g.

b) Show that if g is not constant, then it is also an automorphism of Ω.

Problem 9. Let u be a real valued harmonic function in the punctured unit disc $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

a) Show that there exists precisely one real number A such that $u - A \log |z|$ is the real part of a holomorphic function f (in the punctured disc).

b) Show that if u is bounded, then $A = 0$; moreover, the singularity of f at 0 is removable. (Hence so is that of u!)

Problem 10.

a) Carefully state the following three theorems: the Riemann Mapping Theorem, the Weierstrass Factorization Theorem, and Picard's Little Theorem.

b) Sketch a proof for *one* of the theorems in part a).